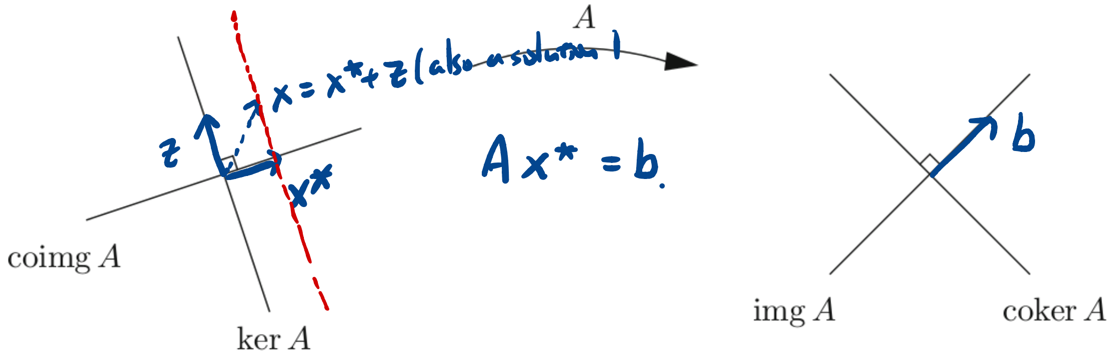
Lecture 27: Quick review from previous lecture

- A compatible linear system $A\mathbf{x} = \mathbf{b}$ with $b \in \operatorname{img} A$ has a unique solution $\mathbf{x}^* \in \operatorname{coimg} A$ satisfying $A\mathbf{x}^* = \mathbf{b}$.

  The general solution is $\mathbf{x} = \mathbf{x}^* + \mathbf{z}$, where $\mathbf{x}^* \in \operatorname{coimg} A$ and $\mathbf{z} \in \ker A$. Then $\mathbf{x}^*$ has the smallest Euclidean norm of all the solutions to $A\mathbf{x} = \mathbf{b}$.



Today we will discuss

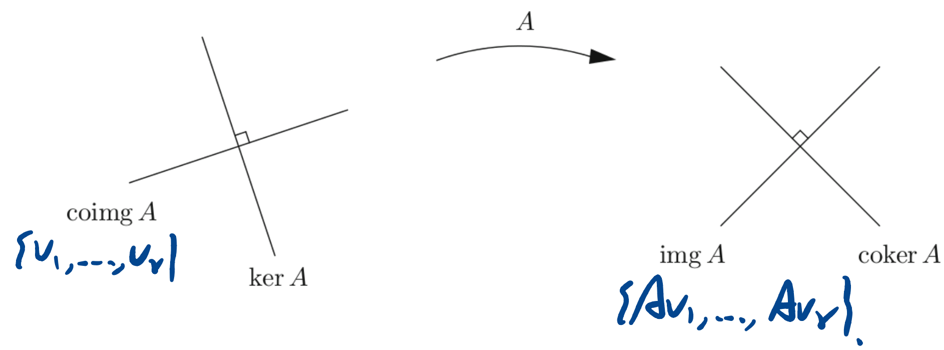
- Section 7.1 Linear Functions

- Lecture will be recorded -

- Exam 2 (next Wednesday 3/31) will cover 2.5, 3.1-3.5, and 4.1-4.4. Instruction and practice exam have been announced on Canvas.

Fact 9: If $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ is a basis of $\operatorname{coimg} A$, then

$$\{A\mathbf{v}_1, \cdots, A\mathbf{v}_r\} \text{ is a basis of } \operatorname{img} A.$$



[Proof] Note that $r = \dim(\operatorname{coimg} A) = \dim(\operatorname{img} A)$. It's sufficient to check $\{A\mathbf{v}_1, \cdots, A\mathbf{v}_r\}$ is linearly independent. Set up $c_1 A\mathbf{v}_1 + \cdots + c_r A\mathbf{v}_r = \mathbf{0}$. Claim that $c_1 = \cdots = c_r = 0$. Now we have $A(c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r) = \mathbf{0}$, and thus

$$c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r \in \ker A \cap (\operatorname{coimg} A)$$

In addition, $c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r$ is also in $\operatorname{coimg} A$. Since $\operatorname{coimg} A \cap \ker A = \{\mathbf{0}\}$, it forces that $c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r = \mathbf{0}$, which implies $c_1 = \cdots = c_r = 0$ due to independence of $\mathbf{v}_i$.

Chapter 7 Linearity

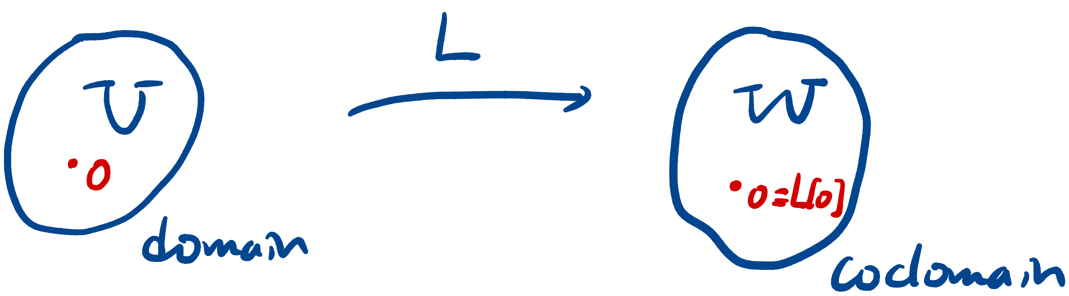
7.1 Linear Functions

Definition: [Linear operators]
If $L : V \to W$ is a mapping between vector spaces V and W, we say that L is **linear** if for all vectors $\mathbf{x}$ and $\mathbf{y}$ in V, and scalars c such that

$$L[c\mathbf{x}] = cL[\mathbf{x}]$$
$$L[\mathbf{x} + \mathbf{y}] = L[\mathbf{x}] + L[\mathbf{y}].$$

We call such a mapping L a **linear operator**. We call V the **domain** for L, and W the **codomain**.

We may also say L is a *linear function*, or a *linear map* (or mapping), or a *linear transformation.* They all refer to the same properties.



Properties:

- For any scalars c and d and any vectors $\mathbf{x}$ and $\mathbf{y}$ in V,

$$L[c\mathbf{x} + d\mathbf{y}] = cL[\mathbf{x}] + dL[\mathbf{y}]$$

- For any scalars $c_1, \cdots, c_n$ and any vectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$ in V, then

$$L[c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n] = c_1L[\mathbf{x}_1] + \cdots + c_nL[\mathbf{x}_n].$$

- $L[\mathbf{0}] = \mathbf{0}$ (the $\mathbf{0}$ on the left is the zero element in V; the $\mathbf{0}$ on the right is the zero element in W).

Example 1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T(x, y) = (x + 2y, 2x - y).$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= A \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then T is a linear mapping.

c, d scalars

$T[c\vec{x} + d\vec{y}]$

$= A[c\vec{x} + d\vec{y}]$

$= cA\vec{x} + dA\vec{y}$

$= cT[\vec{x}] + dT[\vec{y}].$

T is linear map.

Q: What are the linear operators $L : \mathbb{R} \to \mathbb{R}$?

Fact 1: Suppose L is any linear operator $L : \mathbb{R} \to \mathbb{R}$. Then

$$\boxed{L[\mathbf{x}] = a\mathbf{x}} \qquad \text{for a fixed scalar } a.$$

Thus all linear functions from $\mathbb{R} \to \mathbb{R}$ are lines passing through the origins.

[To see this] Observe $L[x] = L[1\, x]$

$x \in \mathbb{R}$ so we can view x as scalar.

$= x\, L[1].$

Let $a = L[1]$. Then

$L[x] = a\, x.$ *

$\boxed{\textbf{Warning}}$: The function $f(x) = ax + b$ is not a linear function unless $b = 0$, even though its graph is also a line; this is because $f(0) = b$, so it doesn't pass through the origin (unless $b = 0$)

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$$

Q: What are the linear operators $L : \mathbb{R}^n \to \mathbb{R}^m$?

Example 2. We can think of A ($m \times n$ matrix) as defining a mapping L from $\mathbb{R}^n$ to $\mathbb{R}^m$, defined by

$$L[\mathbf{v}] = A\mathbf{v} \quad v \in \mathbb{R}^n.$$

The mapping L is **linear** (that is, a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$).

If c, d scalars, Check $L[cv + dw] = c\,L[v] + d\,L[w]$.

Q: Are there any other linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$?
That is, can there be a linear mapping not of the above form, for some matrix A?

> **Fact 2:** Every linear mapping L from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by matrix multiplication,
> $$L[\mathbf{v}] = A\mathbf{v}, \qquad \text{where } A \text{ is an } m \times n \text{ matrix.}$$

[To see this:] Taking $v \in \mathbb{R}^n$, $v = v_1 e_1 + \cdots + v_n e_n$, where $\{e_i\}$ is standard basis of $\mathbb{R}^n$.

$$\begin{aligned} L[v] &= L[v_1 e_1 + \cdots + v_n e_n] \\ &= v_1 L[e_1] + \cdots + v_n L[e_n]. \\ &= \underbrace{\begin{bmatrix} L[e_1] & \cdots & L[e_n] \end{bmatrix}}_{A_{m\times n}} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}. \end{aligned}$$

, $L[e_i]$ = i^{th} column of A.

$C^2 = \{ f \mid f, f', f'' \text{ continuous} \}$.

$C^1 = \{ f \mid f, f' \text{ continuous} \}$

Example 3. Let $C^0([a, b])$ be the vector space of continuous functions on the interval $[a, b]$. Let $C^1([a, b])$ be the space of continuously differentiable functions on $[a, b]$. Check the following functions are linear functions (operators):

1. Define the operator $J : C^0([a, b]) \to C^1([a, b])$ by
$$J[f](x) = \int_a^x f(t)dt, \qquad \text{where } f \in C^0([a, b]).$$

① Check $J[f]$ is C' if $f \in C^0$:

$\frac{d}{dx} J[f](x) = \frac{d}{dx} \int_a^x f(t)\, dt = f(x)$ by Fundamental theory of Calculus,

Then $\frac{d}{dx} J[f] \in C^0$, $J[f] \in C'$. in C^0

② $c, d \in \mathbb{R}$, $f, g \in C^0$: $J[cf + dg] = c\int_a^x f + d\int_a^x g\, dt$

$= c\, J[f] + d\, J[g]$. So J is

a l. op.

2. Now define the operator $D : C^1([a,b]) \to C^0([a,b])$ by

$$D[f](x) = \frac{d}{dx} f(x) = f'(x), \qquad \text{where } f \in C^1([a,b]).$$

① $f \in C^1$. $D[f] = f'(x) \in C^0$.

② $D[cf + dg] = cf' + dg'$

$= c\,D[f] + d\,D[g]$. #

So D is a l. op.

§ **The space of linear functions $\mathcal{L}(V,W)$.**

Let $\mathcal{L}(V,W)$ be the set of all linear functions L mapping from vector space V to vector space W.

Fact 3: $\mathcal{L}(V,W)$ is a vector space.
(see Definition 2.1 in textbook for the definition of a vector space)

Combining with Fact 2, we have

Fact 4: If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, then the space $\mathcal{M}_{m\times n}$ of all $m \times n$ matrices is a vector space, (which is a fact we already knew.)

Example 4. The space of all linear transformations of the plane, $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$, is indeed $\mathcal{M}_{2\times 2}$. And its standard basis are

$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\}$. $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

§ Composition.

Fact 5: If $L : V \to W$ is a linear operator and $M : W \to Z$ is another linear operator, then we can define their **composition** $M \circ L : V \to Z$ by

$$(M \circ L)[\mathbf{v}] = M[L[\mathbf{v}]].$$

Then $(M \circ L)$ is linear.

$$\begin{tikzcd} V \arrow[r, "L{[v]}=Av"] & W \arrow[r, "M{[w]}=Bw"] & Z \\ v \arrow[rr, bend right, "M\circ L{[v]}"'] & L[v] & M[L[v]] \end{tikzcd}$$

Example 5. If $V = \mathbb{R}^n$, $W = \mathbb{R}^m$ and $Z = \mathbb{R}^k$, then

$$L : \mathbb{R}^n \to \mathbb{R}^m, \qquad M : \mathbb{R}^m \to \mathbb{R}^k$$

$$L[\mathbf{v}] = A\mathbf{v}, \qquad M[\mathbf{w}] = B\mathbf{w}$$

for some matrices $A = A_{m\times n}$ and $B = B_{k\times m}$. Consequently, the **composition** $M \circ L$ is given by

$$(M \circ L)[\mathbf{v}] = BA[v].$$

$$\| $$

$$M[L[v]] = B[Av].$$

Example 6. Recall in Example 3. $D : C^2([a, b]) \to C^1([a, b])$ defined by

$$D[f](x) = f'(x).$$

Then $\underline{D \circ D[f]} = D[f'] = f'' \in C^0$ if $f \in C^2$.

$$V \xrightarrow{L} W \xrightarrow{M} V$$

V $\quad M \circ L = I_V \quad$ V

§ Inverses

Definition: Let $L : V \to W$ be a linear operator. If $M : W \to V$ is an operator such that

left inverse of L $\qquad$ right inverse of L

$$M \circ L = I_V, \quad L \circ M = I_W \tag{1}$$

where I_V is the identity map on V, and I_W is the identity map on W. Then we call L is **invertible** and M is the **inverse** of L and write $M = L^{-1}$.

Example 7. If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$,

$$L : \mathbb{R}^n \to \mathbb{R}^m, \qquad M : \mathbb{R}^m \to \mathbb{R}^n$$

$$L[\mathbf{v}] = A\mathbf{v}, \qquad M[\mathbf{w}] = B\mathbf{w}$$

Then the condition (1) is reduced to $\qquad M \circ L[v] = BA\, v.$

$$AB = I_m, \quad BA = I_n.$$

$L \circ M[w] = w.$

Example ~~7~~ 8. Let $J[f](x) = \int_a^x f(t)dt$ be the integration operator, and $D[f](x) = f'(x)$ be differentiation.

(1) Compute $D \circ J$.

(2) Compute $J \circ D$.

To be continued!