Lecture 28: Quick review from previous lecture

- We say $L : V \to W$ is a **linear operator** that maps between vector spaces V and W if for all vectors $\mathbf{x}$ and $\mathbf{y}$ in V, and scalars c such that

$$L[c\mathbf{x}] = cL[\mathbf{x}]$$
$$L[\mathbf{x} + \mathbf{y}] = L[\mathbf{x}] + L[\mathbf{y}].$$

- Let $\mathcal{L}(V, W)$ be the set of all linear functions L mapping from vector space V to vector space W. Then $\mathcal{L}(V, W)$ is a vector space.

- If $L : V \to W$ is a linear operator and $M : W \to Z$ is another linear operator, then we can define their **composition** $M \circ L : V \to Z$ by

$$(M \circ L)[\mathbf{v}] = M[L[\mathbf{v}]].$$

- If $M : W \to V$ is an operator such that

$$V \underset{M}{\overset{L}{\rightleftarrows}} W$$

$$M \circ L = I_V, \quad L \circ M = I_W$$

where I_V is the identity map on V, and I_W is the identity map on W. Then we call L is **invertible** and M is the **inverse** of L and write $M = L^{-1}$.

Today we will discuss

- Section 7.2 linear transformations.

- Lecture will be recorded -

- HW 9 due today at 6pm.
- Exam 2 (next Wednesday) will cover 2.5, 3.1-3.5, and 4.1-4.4.
Instruction and practice exam have been announced on Canvas.

Example 8. Let $J[f](x) = \int_a^x f(t)dt$ be the integration operator, and $D[f](x) = f'(x)$ be differentiation.

$J : C^0 \to C^1$

$D : C^1 \to C^0$

(1) Compute $D \circ J$: $C^0 \to C^0$

(2) Compute $J \circ D$: $C^1 \to C^1$

(1) $f \in C^0$,

$$D \circ J[f] = \frac{d}{dx} \int_a^x f(t)\,dt = f(x).$$

$D \circ J$ = Identity map.

D is left inverse of J.

(2) $f \in C^1$,

$$J \circ D[f] = \int_a^x f'(t)\,dt = f(x) - \underline{f(a)}.$$

If $f(a) = 0$, $J \circ D$ = Identity map, D is right inverse of J.

If $f(a) \neq 0$, D is <u>NOT</u> right inverse of J.

So, when $f(a) = 0$, D is the inverse of J.

7.2 Linear Transformations

Consider a linear function $L : \mathbb{R}^n \to \mathbb{R}^m$. We have known that

> Every linear mapping L from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by matrix multiplication,
> domain codomain
> $$L[\mathbf{x}] = A\mathbf{x}, \qquad \text{where } A \text{ is an } m \times n \text{ matrix.}$$

The following we will see how the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $L[\mathbf{x}] = A\mathbf{x}$ representing the geometrical interpretation.

We need the formulas

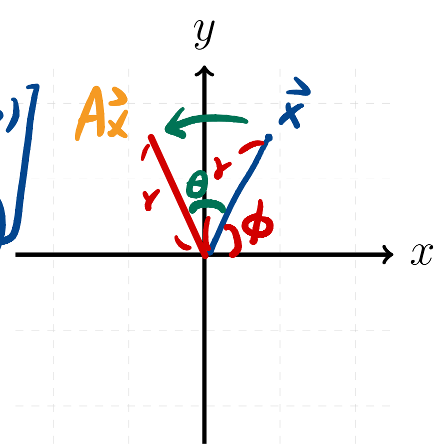
$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi; \qquad \cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

polar coord.

Example 1. If $\mathbf{x} = (r\cos\phi, r\sin\phi)$ is some vector in $\mathbb{R}^2$ (which we are expressing in terms of its polar coordinates), then find $A\mathbf{x}$.

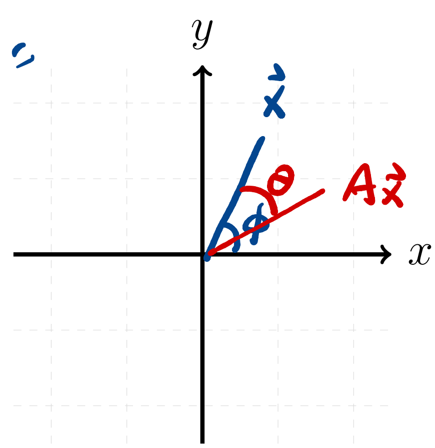
1. $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ (rotation matrix). In addition, $A^TA = AA^T = I$ so A is orthogonal matrix.

$$L[\vec{x}] = A\vec{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix}$$
$$= \begin{bmatrix} r(\cos\theta\cos\phi - \sin\theta\sin\phi) \\ r(\sin\theta\cos\phi + \cos\theta\sin\phi) \end{bmatrix}$$
$$= \begin{bmatrix} r\cos(\theta+\phi) \\ r\sin(\theta+\phi) \end{bmatrix}$$



"rotate counterclockwise by angle θ"

2. $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$



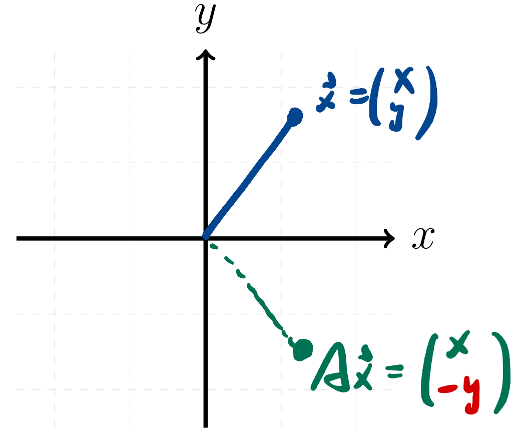
Similarly

$$L[\vec{x}] = A\vec{x} = \begin{bmatrix} r\cos(\phi - \theta) \\ r\sin(\phi - \theta) \end{bmatrix}$$

"rotate clockwise by angle θ".

3. $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (reflection) through x-axis.

$$L\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}.$$



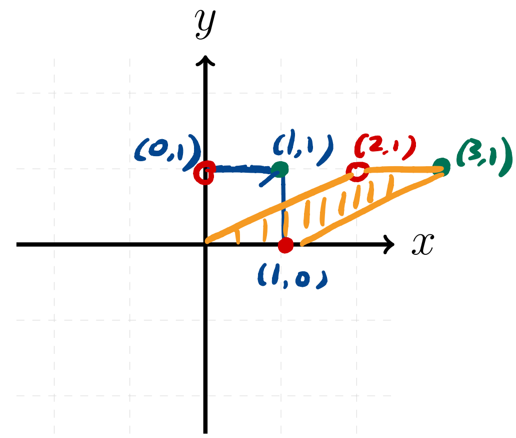
4. $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (shearing) - Shear along the x-axis of magnitude 2

$$L\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ y \end{pmatrix}$$

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$



Example 2. Find the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ which

1. first rotates points counterclockwise about the origin through $\pi/4$;
2. then reflects points through the x-axis.

1. $$M_1[v] = \begin{bmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{bmatrix} v = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{A} v.$$

2. $$M_2[v] = \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}^{B} v.$$

$$L[v] = M_2 \circ M_1[v] = BAv$$
$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} v$$

§ Change of Basis

Let's first consider the following problem.

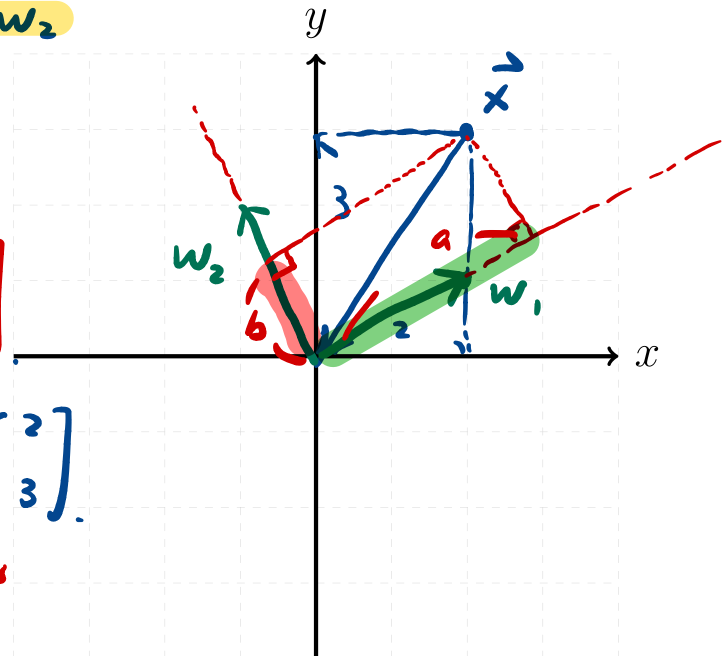
Example 3. Take a point $\vec{\mathbf{x}} = (2,3)$ in $\mathbb{R}^2$, then

$$\vec{\mathbf{x}} = 2\mathbf{e}_1 + 3\mathbf{e}_2, \qquad \text{where } \mathbf{e}_1 = (1,0)^T, \mathbf{e}_2 = (0,1)^T \text{ is the standard basis.}$$

If we take another basis $\mathbf{w}_1 = (2,1)^T$, $\mathbf{w}_2 = (-1,2)^T$ for $\mathbb{R}^2$, then what is the corresponding coordinate $(a,b)^T$ of $\vec{\mathbf{x}}$ to this new basis $\{\mathbf{w}_1, \mathbf{w}_2\}$.

$$\vec{x} = 2e_1 + 3e_2 = a\,w_1 + b\,w_2$$

Find $\begin{pmatrix} a \\ b \end{pmatrix}$.



$$[e_1\ \ e_2]\begin{bmatrix} 2 \\ 3 \end{bmatrix} = [w_1\ \ w_2]\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{[w_1\ \ w_2]}_{\text{New basis}}{}^{-1}\underbrace{[e_1\ e_2]}_{\text{old basis}}\begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7/5 \\ 4/5 \end{bmatrix}.$$ coord. of $\vec{x}$ to $\{w_1, w_2\}$.

1. In $\mathbb{R}^n$, change coordinates from $\mathbf{v}_1, \ldots, \mathbf{v}_n$ to $\mathbf{w}_1, \ldots, \mathbf{w}_n$.

Consider the vector $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ with the coordinate $(x_1, \ldots, x_n)^T$ in a basis $\mathbf{v}_1, \cdots, \mathbf{v}_n$:

$$\vec{\mathbf{x}} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n \qquad \text{in a basis } \mathbf{v}_1, \cdots, \mathbf{v}_n$$

Q: Consider new basis $\mathbf{w}_1, \ldots, \mathbf{w}_n$, how do we find its corresponding coordinate of the same vector $\vec{\mathbf{x}}$?

$$\underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{v}_1, \cdots, \mathbf{v}_n} \longrightarrow \underbrace{\begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}}_{\mathbf{w}_1, \cdots, \mathbf{w}_n}?$$

In other words, finding $(x'_1, \ldots, x'_n)^T$ such that

$$\vec{\mathbf{x}} = x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n = x'_1\mathbf{w}_1 + \ldots + x'_n\mathbf{w}_n.$$

$$\underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{S} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}}_{T} \begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix}$$

$$\begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix} = \underbrace{T^{-1}}_{\text{New}} \underbrace{S}_{\text{old}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Example 4. Let $p(x) = 2x^2 + x + 1$, where

$(2, 1, 1)^T$ is the coordinate of p in the monomial basis $\{x^2, x, 1\}$ of $\mathcal{P}^{(2)}$.

Change the coordinate from the monomial basis

$$\{x^2, x, 1\} \longrightarrow \text{new basis } \{x^2 - x, x - 1, 1\}$$

Find a, b, c so that

$$p(x) = 2x^2 + x + 1 = a(x^2 - x) + b(x-1) + c(1)$$

Old

$$x^2 \xrightarrow{\text{view}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad x \longrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad 1 \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};$$

New

$$x^2 - x \longrightarrow \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad x - 1 \longrightarrow \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad 1 \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = T^{-1} S \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1}}_{\text{New}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{old}} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

$$= \boxed{\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}} \quad \#$$

Check: $2(x^2 - x) + 3(x - 1) + 4 \cdot 1 = \underline{p(x)}$.

2. A linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$, $L[\mathbf{v}] = A\mathbf{v}$.

Consider the vector $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ with the coordinate $(x_1, \ldots, x_n)^T$ in a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$:

$$\vec{\mathbf{x}} = x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n \qquad \text{in a basis } \mathbf{v}_1, \ldots, \mathbf{v}_n$$

$\mathbb{R}^n$ with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ $\longrightarrow$ $\mathbb{R}^n$ with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$

$(x_1, \ldots, x_n)^T$?

$$L[\vec{x}] = \; ? \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Q: How do we find the coordinate $(y_1, \ldots, y_n)^T$ of the vector $L[\vec{\mathbf{x}}]$ to the basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$?

$$\vec{\mathbf{x}} = x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n \longrightarrow \vec{\mathbf{y}} = L[\vec{\mathbf{x}}] = y_1\mathbf{v}_1 + \ldots + y_n\mathbf{v}_n$$

① $L[\vec{x}] = L[x_1 v_1 + \cdots + x_n v_n] = x_1 A v_1 + \cdots + x_n A v_n$

$$= A \underbrace{[v_1 \cdots v_n]}_{S} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

② $\vec{y} = L[\vec{x}] = y_1 v_1 + \cdots + y_n v_n = \underbrace{[v_1 \cdots v_n]}_{S} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

① = ② yields

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \underbrace{S^{-1}AS}_{= B} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

$B = S^{-1}AS$ is the matrix representation of L ~~is~~ in bases $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

NOTE: A is matrix representation of L in standard basis $\{e_i\}$.

Poll Question 1: Let $L : V \to W$ be a linear operator. For $\mathbf{v}, \mathbf{w} \in V$ and scalars c, d, which property is true?

$A)$ $L[\mathbf{v} + \mathbf{w}] = L[\mathbf{v}] + L[\mathbf{w}]$

$B)$ $L[-9\mathbf{v}] = 9L[\mathbf{v}]$

-9