Lecture 29: Quick review from previous lecture

- In $\mathbb{R}^n$, for vector $\vec{\mathbf{x}}$, change coordinates from $\mathbf{v}_1, \ldots, \mathbf{v}_n$ to $\mathbf{w}_1, \ldots, \mathbf{w}_n$:

$\vec{x} = x_1 v_1 + \cdots + x_n v_n$

$$\underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\mathbf{v}_1,\cdots,\mathbf{v}_n} \longrightarrow \underbrace{\begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}}_{\mathbf{w}_1,\cdots,\mathbf{w}_n}, \qquad \text{is} \qquad \underbrace{\begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}}_{\text{New}} = \underbrace{[\mathbf{w}_1 \ldots \mathbf{w}_n]^{-1}}_{\text{New}} \underbrace{[\mathbf{v}_1 \ldots \mathbf{v}_n]}_{\text{old}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

- Consider a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$, $L[\mathbf{x}] = A\mathbf{x}$.

  $\mathbb{R}^n$ with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ $\longrightarrow$ $\mathbb{R}^n$ with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$

  $\vec{\mathbf{x}}, (x_1, \ldots, x_n)^T$ input $\qquad$ $L[\vec{\mathbf{x}}], (y_1, \ldots, y_n)^T$ output

  Then

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \underbrace{S^{-1}AS}_{B} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

  where $S = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$. The **matrix representation** of L in these bases is $B = S^{-1}AS$.

Today we will discuss

- Section 7.2 linear transformations.

- Lecture will be recorded -

- Exam 2 (**This Wednesday**) will cover 2.5, 3.1-3.5, and 4.1-4.4. Instruction and practice exam have been announced on Canvas.
- Exam time is 50 mins with additional 10 mins for scanning and submission to Canvas. This is **a closed book exam** and everyone needs to **Open Camera**.

3. A linear operator $L: \mathbb{R}^n \to \mathbb{R}^m$, $L[\mathbf{v}] = A\mathbf{v}$.

Consider the vector $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ with the coordinate $(x_1, \ldots, x_n)^T$ in a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$:

$$\vec{\mathbf{x}} = x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n \quad \text{in a basis } \mathbf{v}_1, \ldots, \mathbf{v}_n$$

$\mathbb{R}^n$ with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ $\xrightarrow{L}$ $\mathbb{R}^m$ with basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$

$\vec{x}$ $\quad (x_1, \ldots, x_n)^T$ $\qquad$ output $L[\vec{x}]$ $\quad ?,\ \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$.

Q: How do we find the coordinate $(y_1, \ldots, y_m)^T$ of the vector $L[\vec{\mathbf{x}}]$ to the basis $\mathbf{w}_1, \ldots, \mathbf{w}_m$?

$$\vec{\mathbf{x}} = x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n \longrightarrow \vec{\mathbf{y}} = L[\vec{\mathbf{x}}] = y_1\mathbf{w}_1 + \ldots + y_m\mathbf{w}_m$$

① $L[\vec{x}] = L[x_1v_1 + \cdots + x_nv_n] = x_1L[v_1] + \cdots + x_nL[v_n]$

$$= x_1 Av_1 + \cdots + x_n Av_n = A\underbrace{[v_1 \cdots v_n]}_{S}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

② $y = L[\vec{x}] = y_1w_1 + \cdots + y_mw_m$

$$= \underbrace{[w_1 \cdots w_m]}_{T}\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}.$$

① = ② yields

$$T\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = AS\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Then
$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = T^{-1}AS\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

$B = T^{-1}AS$ is the matrix representation of L in bases $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_m$, where $T = [\mathbf{w}_1, \ldots, \mathbf{w}_m]$ and $S = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$.

✓ One reason for changing basis is that some coordinate systems are better-adapted for a particular operator L than others. $L:\mathbb{R}^2 \to \mathbb{R}^2$

Example 5. Consider the operator

$$L\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \overbrace{\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}}^{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x - y \\ 5y - x \end{pmatrix}$$

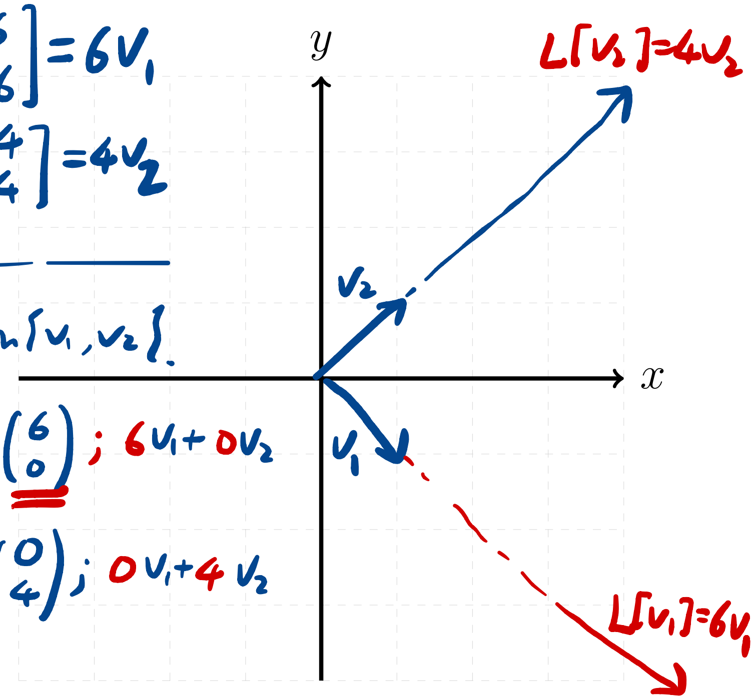
Consider a new basis $\mathbf{v}_1 = (1, -1)^T$, $\mathbf{v}_2 = (1, 1)^T$ for both domain and codomain. The matrix representation of L in the basis $\mathbf{v}_1$, $\mathbf{v}_2$ is

$$B = S^{-1}AS = [\mathbf{v}_1, \mathbf{v}_2]^{-1} \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} [\mathbf{v}_1, \mathbf{v}_2]$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}} = B$$

$L[v_1] = Av_1 = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix} = 6v_1$

$L[v_2] = Av_2 = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4v_2$



coord. in $\{v_1, v_2\}$ $\xrightarrow{L}$ coord. in $\{v_1, v_2\}$.

$1v_1 + 0v_2$; $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\xrightarrow{B}$ $B\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$; $6v_1 + 0v_2$

$0v_1 + 1v_2$; $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\xrightarrow{B}$ $B\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$; $0v_1 + 4v_2$

- In other words, if $\mathbf{x} = a\mathbf{v}_1 + b\mathbf{v}_2$, then $L[\mathbf{x}] = 6a\mathbf{v}_1 + 4b\mathbf{v}_2$. So L scales along the direction of $\mathbf{v}_1$ by 6, and scales along the direction of $\mathbf{v}_2$ by 4.
- The simple geometry of L is only revealed by the new basis; it is not apparent from the matrix in the standard basis, $\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$.

Example 6. Suppose we have the operator $L : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$L\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = \begin{pmatrix} x+y \\ y+z \end{pmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ represents L in the standard basis. $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$ & $\{e_1, e_2\}$ for $\mathbb{R}^2$.

Q: Consider the basis

$$\mathbf{v}_1 = (1,0,1)^T, \quad \mathbf{v}_2 = (1,-1,0)^T, \quad \mathbf{v}_3 = (0,1,-1)^T \quad \text{of } \mathbb{R}^3$$

and

$$\mathbf{w}_1 = (1,1)^T, \quad \mathbf{w}_2 = (-1,1)^T \quad \text{of } \mathbb{R}^2.$$

What would be the matrix representation of L in these bases?

$$\mathbb{R}^3, \{v_i\} \xrightarrow{L} \mathbb{R}^2, \{w_i\}.$$

matrix representation of L

$$B = T^{-1} A S, \quad \text{where } T = [w_1\ w_2], \quad S = [v_1\ v_2\ v_3].$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix}. \quad \#$$

Observation: coord. in $\{v_i\}$ $\xrightarrow{L}$ coord. in $\{w_i\}$.

$1v_1 + 0v_2 + 0v_3$; $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{B} B\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $1w_1 + 0w_2$.

$0v_1 + 1v_2 + 0v_3$; $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{B} B\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix}$; $-\frac{1}{2}w_1 + (-\frac{1}{2})w_2$

So applying B to the coefficients of a vector $\mathbf{v}$ in the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ returns the coefficients of $L[\mathbf{v}]$ in the basis $\mathbf{w}_1, \mathbf{w}_2$.

Example 6. Recall that $\mathcal{P}^{(2)}$ is the vector space of polynomials of degree ≤ 2; $\mathcal{P}^{(1)}$ is the vector space of polynomials of degree ≤ 1.

Let the linear operator $L : \mathcal{P}^{(2)} \to \mathcal{P}^{(1)}$ satisfy

$$L[p](x) = p'(x).$$

(1) Consider the basis $\{x^2, x, 1\}$ for $\mathcal{P}^{(2)}$ and the basis for $\{x, 1\}$ for $\mathcal{P}^{(1)}$. Find the matrix representation of L in these bases.[1]

Taking any $p(x) = ax^2 + bx + c$ in $P^{(2)}$.

$$L[ax^2 + bx + c] = 2ax + b.$$

$\downarrow$ view $\qquad\qquad$ $\downarrow$ view

$$\begin{array}{c} x^2 \to \\ x \to \\ 1 \to \end{array} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad\qquad \begin{pmatrix} 2a \\ b \end{pmatrix}.$$

$$L\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right] = \begin{pmatrix} 2a \\ b \end{pmatrix} = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Sol: A. #.

(2) We instead use the basis $\{x^2 - x, x - 1, 1\}$ for $\mathcal{P}^{(2)}$ and the basis $\{2x, 1\}$ for $\mathcal{P}^{(1)}$. Find the matrix representation of L in these bases.

$P^{(2)} \longrightarrow P^{(1)}$

$x^2 - x$, $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\qquad$ $2x$, $w_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$x - 1$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ $\qquad$ 1, $w_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

1, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$B = T^{-1} A S = [w_1\ w_2]^{-1} A [v_1\ v_2\ v_3].$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}. \quad \#$$

[1] *Note that the matrix A that represents L depends on the choice of basis for $\mathcal{P}^{(2)}$ and $\mathcal{P}^{(1)}$!

§ Canonical form of the operator L.

We take any matrix $A = A_{m\times n}$ with $\mathsf{rank} A = r$. Let $L[\mathbf{x}] = A\mathbf{x}$. Consider "a suitable choice of bases" for domain and codomain:

$$\mathbb{R}^n \text{ with basis} \quad \xrightarrow{L[x]=Ax} \quad \mathbb{R}^m \text{ with basis}$$

$$\{\underbrace{\mathbf{v}_1,\dots,\mathbf{v}_r}_{coimg A}, \underbrace{\mathbf{v}_{r+1},\dots\mathbf{v}_n}_{ker A}\} \longrightarrow \{\underbrace{\mathbf{w}_1,\dots,\mathbf{w}_r}_{img A}, \underbrace{\mathbf{w}_{r+1},\dots\mathbf{w}_m}_{coker A}\}$$

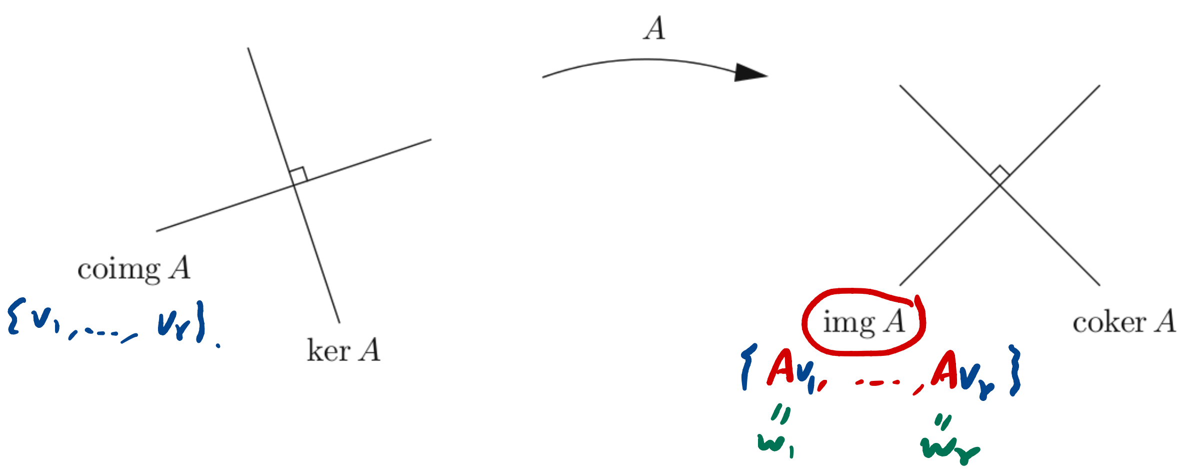
Then the matrix representation of L in bases $\mathbf{v}_1,\dots,\mathbf{v}_n$ and $\mathbf{w}_1,\dots,\mathbf{w}_m$ is

$$B = T^{-1}AS = \begin{bmatrix} I_r & O_{r\times(n-r)} \\ O_{(m-r)\times r} & O_{(m-r)\times(n-r)} \end{bmatrix}_{m\times n} \qquad \text{(Canonical Form)},$$

where $T = [\mathbf{w}_1,\dots,\mathbf{w}_m]$ and $S = [\mathbf{v}_1,\dots,\mathbf{v}_n]$ with $A\mathbf{v}_j = \mathbf{w}_j$ for $j = 1, \dots, r$.

[To see this:] Take $\mathbf{v}_1,\dots,\mathbf{v}_r$ a basis for coimg A, and $\mathbf{v}_{r+1},\dots,\mathbf{v}_n$ a basis for ker A. So $\mathbf{v}_1,\dots,\mathbf{v}_n$ are a basis for $\mathbb{R}^n$.

As we've known, $\{\mathbf{w}_1 = A\mathbf{v}_1,\dots,\mathbf{w}_r = A\mathbf{v}_r\}$ will be a basis for img A.



Take $\mathbf{w}_{r+1},\dots,\mathbf{w}_m$ to be any basis for coker A. Then

$$\{\mathbf{w}_1,\dots,\mathbf{w}_r, \mathbf{w}_{r+1},\cdots\mathbf{w}_m\} \quad \text{is a basis for } \mathbb{R}^m.$$

$Av_i = w_i, \quad 1 \le i \le r$.

$Av_j = 0, \quad r+1 \le j \le n$ since $v_j \in \ker A$.

$$A\underbrace{[v_1 \cdots v_r \; v_{r+1} \cdots v_n]}_{S} = [w_1 \cdots w_r \; 0 \cdots 0]$$

$$= \underbrace{[w_1 \cdots w_r \; w_{r+1} \cdots w_n]}_{T} \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right].$$

$$\Rightarrow B = T^{-1}AS = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$$

Remark: In other words, the top r-by-r block is the identity, and everywhere else it has 0. This is the **canonical form** of the operator L, that depends "only" on its rank.

Example 7. Let the operator $L[\mathbf{x}] = A\mathbf{x}$, where $L: \mathbb{R}^2 \to \mathbb{R}^3$.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{pmatrix}.$$

Find the **canonical form** of the operator $L[\mathbf{x}] = A\mathbf{x}$.

$$A \longrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{rank } A = 1.$$

Canonical form of L is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$$\mathbb{R}^2 \xrightarrow{\ L\ } \mathbb{R}^3$$

Find a basis for coimg A: $\left\{ v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

" img A: $\left\{ w_1 = A v_1 = \begin{pmatrix} 5 \\ 10 \\ -5 \end{pmatrix} \right\}$.

" ker A: $\left\{ v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$.

" coker A: $\left\{ w_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \ w_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$