#### Lecture 3: Quick review from previous lecture

- We have learned how to use Gaussian elimination to solve a linear system
   Ax = b when A is regular, that means that a matrix only has nonzero
   pivots. (odding/substancting one row to/from other row)
- We have shown that such regular matrix A can be factored as

$$A = LU,$$

where U is upper triangular and L is lower triangular. Furthermore, L has 1's on its main diagonal, and U has non-zero elements on its main diagonal (the pivots of A).

Today we will

- continue discuss Sec. 1.3 Gaussian Elimination
- discuss Sec. 1.4 Pivoting and Permutations

### - Lecture will be recorded -

• The first homework is due this Friday (1/29) at 6pm.

**Example 4:** Find LU factorization of the matrix

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -5 & 3 \\ 1 & 4 & -2 \end{pmatrix}, L = I_{3}$$

$$A \xrightarrow{3-40} \begin{pmatrix} 0 & -2 & 1 \\ 0 & 3 & -1 \\ 1 & 4 & (-2) \end{pmatrix}, L = E_{1}^{4} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

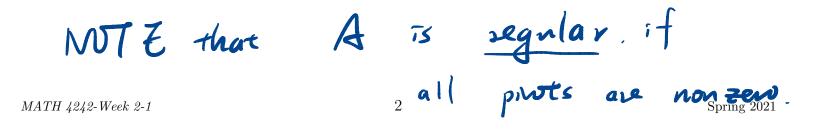
$$E_{1} = \begin{pmatrix} -4 & 1 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, L = E_{1}^{-1} E_{2}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_{2} = \begin{pmatrix} -2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & -3 \end{pmatrix}, L = E_{1}^{-1} E_{2}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$E_{3} = \begin{pmatrix} -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix} = I , L = E_{1}^{-1} E_{2}^{-1} E_{1}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$E_{3} = \begin{pmatrix} -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix} = I , L = E_{1}^{-1} E_{2}^{-1} E_{1}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$E_{3} = \begin{pmatrix} -2 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & -3 \end{pmatrix} = I , L = E_{1}^{-1} E_{2}^{-1} E_{1}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



§ Use "LU factorization" to solve a linear system  $A\mathbf{x} = \mathbf{b}$ . We do this by solving 2 linear systems:

$$A \times = b = \Rightarrow L \amalg x = b.$$

$$1^{st} \text{ system} : \left[ \bigsqcup_{y = b} \right] \text{ solve for } y = b \text{ using forward substitutivis}$$

$$2^{nd} \text{ system} = \left[ \bigsqcup_{x = y} \right] \text{ solve for } \times b \text{ using back substitutivis}$$

**Example 5:** Consider the same matrix A as in Example 4,

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 4 & -5 & 3 \\ 1 & 4 & -2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}.$$

Solve  $A\mathbf{x} = \mathbf{b}$  by using LU factorization.

### **1.4 Pivoting and Permutations**

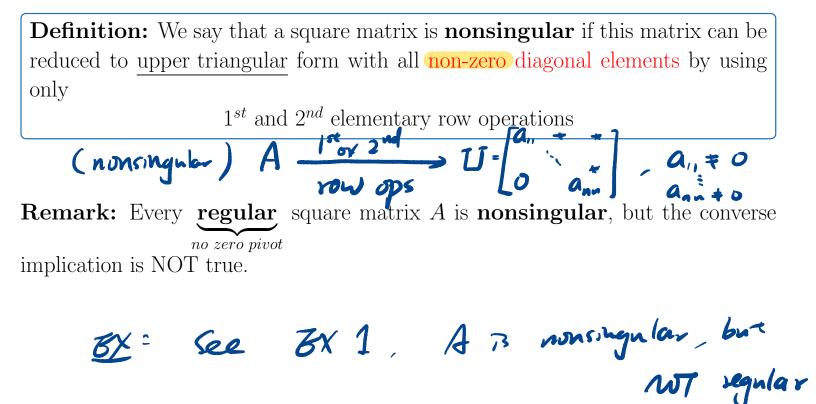
From the following example, we will learn how to handle the situation, where some pivot of the matrix A is zero when we perform Gaussian Elimination. Example 1. Solve the linear system  $A\mathbf{x} = \mathbf{b}$ , where

The operation of <u>permuting two rows of</u> the matrix (or equivalently permuting the order of equations), is called **pivoting**.

From now on,

when we refer to "Elementary Row Operation": It includes

- the 1<sup>st</sup> type of elementary row operation: add/subtracting a multiple of one row to/from another row
- the 2nd type of elementary row operation: pivoting ( switch /permite)



**Definition:** A matrix that is not nonsingular is called **singular**.

**Definition:** A **permutation matrix** is a matrix obtained from the identity matrix  $I_n$  by any combination of row interchanges.

# Example 2:

(1) Write down the 3-by-3 permutation matrix that swaps the order of rows 2 and 3.

$$I_{3} \xrightarrow{\text{mer}} \left(\begin{array}{c} 0 & 0 & 1 \\ 0 & 0 \end{array}\right) = P$$

$$2) \text{ Let } B = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 5 \\ 3 & 4 & 7 \end{pmatrix}$$

$$(a) PB = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 5 \\ 3 & 4 & 7 \end{pmatrix}$$

$$(b) \text{ Is } B \text{ regular?} \\ B \xrightarrow{\textbf{O}-20} \\ \overrightarrow{\textbf{O}-10} \\ \overrightarrow{\textbf{O}-10}$$

Example 3: Same matrix *B* as in Example 2. Let *P* be a permutation matrix. Suppose that  $PB = \begin{pmatrix} 3 & 4 & 7 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \\ 2 & 2 & 5 \\ 2 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 &$   $\checkmark$  However, in this case P is NOT considered an "elementary matrix", since this permutation is NOT simply swapping two rows one time.

**Remark:** Multiply two or more permutation matrices, we obtain another permutation matrix. EX. If  $P_1$ ,  $P_2$  are permutation matrices, so is  $P_1P_2$ .

## § The permuted LU factorization

Every **nonsingular** matrix A can be reduced to <u>upper triangular matrix</u> by applying elementary row operator of type 1 and type 2:

Fact 1. A is square matrix. The following are equivalent:
1. A is nonsingular.
2. A has a permuted LU factorization: PA = LU., P permutation. matrix
\* If matrix A is regular, then permutation matrix P above is simply identity matrix (P = I) since we do not need to do any row switchings. A = LU

Next we illustrate the **general method to construct LU factorization** of a matrix A by doing the following example: We will systematically build L, U and P.

Τ.,

**Example 4.** Find LU factorization of the matrix

**Remark:** When A is not regular, performing  $2^{nd}$  type of elementary row operation (permuting rows) indeed can give: *PA is regular (has all nonzero pivot)*. Then we can find its *LU* factorization, namely,

$$PA = LU$$