Lecture 3: Quick review from previous lecture

- We have learned how to use Gaussian elimination to solve a linear system $A\mathbf{x} = \mathbf{b}$ when A is **regular**, that means that a matrix only has nonzero pivots. (adding/substracting " one row to/from other row)
- We have shown that such regular matrix A can be factored as

$$A = LU,$$

where U is upper triangular and L is lower triangular.
Furthermore, L has 1's on its main diagonal, and U has $\boxed{\text{non-zero}}$ elements on its main diagonal (the pivots of A).

Today we will

- continue discuss Sec. 1.3 Gaussian Elimination
- discuss Sec. 1.4 Pivoting and Permutations

- Lecture will be recorded -

- The first homework is due this Friday (1/29) at 6pm.

Example 4: Find LU factorization of the matrix

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 4 & -5 & 3 \\ 1 & 4 & -2 \end{pmatrix}, \quad L = I_3.$$

$$A \xrightarrow{②-4①} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & -1 \\ 1 & 4 & -2 \end{pmatrix}, \quad L = E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

pivot

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{③-①} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & -1 \\ 0 & 6 & -3 \end{pmatrix}, \quad L = E_1^{-1}E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{③-2②} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & -1 \end{pmatrix} = U, \quad L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

Then $A = LU$.

NOTE that A is regular if all pivots are nonzero.

§ **Use "LU factorization" to solve a linear system $A\mathbf{x} = \mathbf{b}$.**

We do this by solving 2 linear systems:

$$A\,x = b. \Rightarrow L\underbrace{Ux}_{y} = b.$$

1^{st} system: $\boxed{L\,y = b}$. Solve for y by using "forward substitution"

$$\begin{bmatrix} 1 & & 0 \\ * & \ddots & \\ * & * & 1 \end{bmatrix}[y] = [b]$$

2^{nd} system: $\boxed{U\,x = y}$. Solve for x by using "back substitution".

$$[\,\nabla\,]\,x = y$$

Example 5: Consider the same matrix A as in Example 4,

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 4 & -5 & 3 \\ 1 & 4 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}.$$

Solve $A\mathbf{x} = \mathbf{b}$ by using LU factorization.

1. $\boxed{L\,y = b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}.$$

① $\underline{y_1 = 1}$; ② $4 + y_2 = 6 \Rightarrow \underline{y_2 = 2}$.

③ $1 + 2(2) + y_3 = 3 \Rightarrow \underline{y_3 = -2}$

2. $\boxed{U\,x = y}$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

③ $x_3 = \boxed{2}$.

② $3x_2 - x_3 = 2 \Rightarrow x_2 = \boxed{4/3}$

① $x_1 - 2x_2 + x_3 = 1 \Rightarrow x_1 = \boxed{5/3}$ #

1.4 Pivoting and Permutations

From the following example, we will learn how to handle the situation, where some pivot of the matrix A is zero when we perform Gaussian Elimination.

Example 1. Solve the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 3 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$$

$$(A\,|\,b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 2 & 4 \\ 1 & 3 & 1 & 5 \end{array}\right)$$

A is NOT regular

A is nonsingular.

$$\xrightarrow[③ - ①]{② - 2①} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & -4 & 2 \\ 0 & 1 & -2 & 4 \end{array}\right)$$

$$\xrightarrow{\text{switch } ② ③} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & -4 & 2 \end{array}\right)$$

upper triangular form.

By back substitution, we can solve x, y, z.

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The operation of permuting two rows of the matrix (or equivalently permuting the order of equations), is called **pivoting**.

From now on,

when we refer to **"Elementary Row Operation"**: It includes

- **the 1^{st} type of elementary row operation:** add/subtracting a multiple of one row to/from another row
- **the 2^{nd} type of elementary row operation:** pivoting (switch /permute).

Definition: We say that a square matrix is **nonsingular** if this matrix can be reduced to upper triangular form with all non-zero diagonal elements by using only

1^{st} and 2^{nd} elementary row operations

$$(\text{nonsingular})\ A \xrightarrow[\text{row ops}]{1^{st}\text{ or }2^{nd}} U = \begin{bmatrix} a_{11} & * & * \\ & \ddots & * \\ 0 & & a_{nn} \end{bmatrix}, \quad a_{11} \neq 0, \ \vdots, \ a_{nn} \neq 0$$

Remark: Every $\underbrace{\textbf{regular}}_{\textit{no zero pivot}}$ square matrix A is **nonsingular**, but the converse implication is NOT true.

$\underline{\text{EX}}$: See EX 1, A is nonsingular, but NOT regular

Definition: A matrix that is not nonsingular is called **singular**.

> **Definition:** A **permutation matrix** is a matrix obtained from the identity matrix I_n by any combination of row interchanges.

Example 2:

(1) Write down the 3-by-3 permutation matrix that swaps the order of rows 2 and 3.

$$I_3 \xrightarrow[\text{2 3}]{\text{swap}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = P.$$

(2) Let $B = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 5 \\ 3 & 4 & 7 \end{pmatrix}$

(a) $PB =$

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & 4 & 7 \\ 2 & 2 & 5 \end{pmatrix}.$$

(b) Is B regular?

$$B \xrightarrow[3-3(1)]{2-2(1)} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

. B is NOT regular.

PB is regular. $\Longrightarrow$ $\boxed{PB = LU.}$

Example 3: Same matrix B as in Example 2. Let P be a permutation matrix. Suppose that $PB = \begin{pmatrix} 3 & 4 & 7 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$. Find P.

$$P_{13}\, P_{12} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. = P$$

.

✓ However, in this case P is NOT considered an "elementary matrix", since this permutation is NOT simply swapping two rows one time.

> **Remark:** Multiply two or more permutation matrices, we obtain another permutation matrix. EX. If P_1, P_2 are permutation matrices, so is P_1P_2.

§ The permuted LU factorization

Every **nonsingular** matrix A can be reduced to upper triangular matrix by applying elementary row operator of type 1 and type 2:

> **Fact 1.** *A is (n×n) square matrix. The following are equivalent:*
>
> 1. *A is* **nonsingular**.
> 2. *A has a permuted LU factorization:* $\boxed{PA = LU}$. P permutation matrix

* If matrix A is **regular**, then permutation matrix P above is simply identity matrix ($P = I$) since we do not need to do any row switchings. $\boxed{A = LU}$

I_n

Next we illustrate the **general method to construct LU factorization** of a matrix A by doing the following example:
We will systematically build L, U and P.

Example 4. Find LU factorization of the matrix

$$A = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 3 & -3 \\ -2 & -6 & -2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}, \quad L = I_4, \quad P = I_4.$$

$$A \xrightarrow[\textcircled{4}-\textcircled{1}]{\textcircled{2}-2\textcircled{1},\ \textcircled{3}+2\textcircled{1}} \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad P = I_4$$

$$\xrightarrow{\textcircled{2}\leftrightarrow\textcircled{4}} \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & -7 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

$$\xrightarrow{\textcircled{3}\leftrightarrow\textcircled{4}} \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 5 \end{pmatrix} = U, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then $PA = LU$

Remark: When A is not regular, performing 2^{nd} type of elementary row operation (permuting rows) indeed can give: *PA is regular (has all nonzero pivot).* Then we can find its LU factorization, namely,

$$\boxed{PA = LU}$$