Lecture 30: Quick review from previous lecture

- Consider a linear operator $L : \mathbb{R}^n \to \mathbb{R}^m$, $L[\mathbf{x}] = A\mathbf{x}$. , $A_{m\times n}$.

  $\mathbb{R}^n$ with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ $\longrightarrow$ $\mathbb{R}^m$ with basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$

  $\vec{\mathbf{x}}, (x_1, \ldots, x_n)^T$ $\qquad$ $L[\vec{\mathbf{x}}], (y_1, \ldots, y_m)^T$

  Then
  $$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \underbrace{T^{-1}AS}_{B} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$
  where $T = [\mathbf{w}_1, \ldots, \mathbf{w}_m]$ and $S = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$. The **matrix representation** of L in these bases is $B = T^{-1}AS$.

Today we will discuss

- Section 8.2 Eigenvalues and eigenfunctions.

- Lecture will be recorded -

- HW 10 due ~~today at 6pm.~~ next Monday at 6pm.
- The solutions, statistic, and grade for Exam 2 were posted on Canvas.
- The University provides free **peer tutor service**, which can be found in `https://www.lib.umn.edu/smart` (SMART Learning Commons)

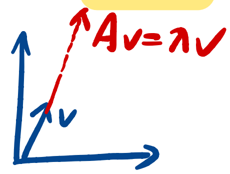
Chapter 8 Eigenvalues and Singular Values

We will discuss 8.2, 8.3, 8.5, and 8.7.

8.2 Eigenvalues and Eigenvectors

As we will see, eigenvectors are a natural basis for expressing the action of symmetric linear operators.

Definition: If $A = A_{n\times n}$ is a square matrix, we say that a scalar λ is an **eigenvalue** of A if there is a non-zero vector $\mathbf{v} \neq \mathbf{0}$ satisfying



$$\boxed{A\mathbf{v} = \lambda\mathbf{v}}$$

If λ is an eigenvalue, we say a vector $\mathbf{v} \neq \mathbf{0}$ satisfying $A\mathbf{v} = \lambda\mathbf{v}$ is an **eigenvector**.

***Important:** The zero vector $\mathbf{0}$ is **not** allowed to be an eigenvector, by definition.

Here the action of a matrix A on the eigenvector mimic scalar multiplication.

Properties:

1) In geometric terms, the eigenvectors of A are those vectors that are stretched or scaled by A. See also Lecture 29 Example 5.

2) The eigenvalue λ is the amount by which the eigenvector $\mathbf{v}$ is stretched.

3) Note that even though $\mathbf{v} \neq \mathbf{0}$, we may have $\lambda = 0$. $Av = 0v, \ v \neq 0.$

Goal:

$$A \longrightarrow \begin{pmatrix} * & 0 & \dots & 0 \\ 0 & * & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & * \end{pmatrix}$$

§ How to find eigenvalues and eigenvectors.

Let's rewrite the equations $A\mathbf{v} = \lambda\mathbf{v}$ into

$$(A - \lambda I)\mathbf{v} = \mathbf{0}, \qquad \text{where } I \text{ is the identity matrix.} \tag{1}$$

Clearly, it is a homogeneous linear system, and thus $\mathbf{v} = \mathbf{0}$ is a solution of (1).

Q: How to find its nonzero solutions (eigenvectors $\mathbf{v}$)?

In other words, the eigenvectors $\mathbf{v}$ with eigenvalue λ are the non-zero vectors in the kernel of $A - \lambda I$.

$v \neq 0$, $v \in \ker(A - \lambda I)$.

Thus, we have the following fact.

Fact 1: A scalar λ is an eigenvalue of $n \times n$ matrix A if and only if $A - \lambda I$ is singular ($\cancel{\operatorname{rank} A < n}$).

$\operatorname{rank}(A - \lambda I) < n$. → NOT invertible.

From Fact 1, we immediately have

Fact 2: A scalar λ is an eigenvalue of $n \times n$ matrix A if and only if λ is a solution to the **characteristic equation**

$$\det(A - \lambda I) = 0.$$

Summary:

$$\begin{aligned}
\lambda \text{ is an eigenvalue} &\iff \text{there is a nonzero vector } \mathbf{v} \text{ so that } A\mathbf{v} - \lambda\mathbf{v} = 0 \\
&\iff A - \lambda I \text{ is singular} \\
&\iff \det(A - \lambda I) = 0.
\end{aligned}$$

We define **characteristic polynomial of** A by

$$p_A(\lambda) = \det(A - \lambda I),$$, polynomial with degree n. if $A_{n\times n}$

the eigenvalues of A are the **roots** of $p_A(\lambda)$, i.e. the values λ at which $p_A(\lambda) = 0$.

Example 1. Let $A = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$. Find eigenvalues and eigenvectors.

① Set up $\det(A - \lambda I) = 0$.

$0 = \det\begin{pmatrix} 1-\lambda & -1 \\ -2 & 0-\lambda \end{pmatrix} = (1-\lambda)(-\lambda) - 2$

$= \lambda^2 - \lambda - 2$

$= (\lambda - 2)(\lambda + 1)$

$\begin{matrix} 2 & -2 \\ -1 & 1 \end{matrix}$

$\lambda = 2, -1$ are eigenvalues.

② Find eigenvectors ($\overset{\neq 0}{v} \in \ker(A - \lambda I)$)

$\underline{\lambda = 2}$: $A - 2I = \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix} \xrightarrow{②-2①} \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$.

$x = -y$. $\ker(A - 2I) = \left\{ \begin{pmatrix} -y \\ y \end{pmatrix} \middle| y \in \mathbb{R} \right\}$.

Let $y = 1$, $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ eigenvector corresponding to $\lambda = 2$.

$\underline{\lambda = -1}$: $A + I = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}$.

$v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ eigenvector corresponding to $\lambda = -1$.

or $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\begin{pmatrix} -7 \\ -14 \end{pmatrix}$ ... are also okay.

Remark: If $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue λ, then so is every nonzero scalar multiple of $\mathbf{v}$, that is, $c\mathbf{v}$ for scalar $\underline{c \neq 0}$.

Example 2. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & -1 & 5 \end{pmatrix}$. Find its eigenvalues and eigenvectors.

① Set up $\det(A - \lambda I) = 0$

$$0 = \det \begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 5-\lambda & -1 \\ 0 & -1 & 5-\lambda \end{pmatrix} = (2-\lambda)\det\begin{pmatrix} 5-\lambda & -1 \\ -1 & 5-\lambda \end{pmatrix} + 0 + 0.$$

$$= (2-\lambda)\left[(5-\lambda)^2 - 1\right]$$

$$= (2-\lambda)(\lambda^2 - 10\lambda + 24)$$

$$= (2-\lambda)(\lambda-4)(\lambda-6)$$

$$\begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 5-\lambda & -1 \\ 0 & -1 & 5-\lambda \end{pmatrix} \xrightarrow{③ + \frac{1}{5-\lambda}②} \begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 5-\lambda & -1 \\ 0 & 0 & 5-\lambda-\frac{1}{5-\lambda} \end{pmatrix}$$

$$\det(A-\lambda I) = (2-\lambda)(5-\lambda)\left(5-\lambda - \frac{1}{5-\lambda}\right)$$

$$= (2-\lambda)(5-\cancel{\lambda})\left(\frac{(5-\lambda)^2 - 1}{\cancel{(5-\lambda)}}\right)$$

$\lambda = 6, 4, 2$ are eigenvalues.

② Find eigenvectors ($v \in \ker(A - \lambda I)$).

$\lambda = 6$: $A - 6I = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{③-②} \begin{pmatrix} -4 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$

$y = -z;\quad x = 0$

$\ker(A - 6I) = \left\{ \begin{pmatrix} 0 \\ -z \\ z \end{pmatrix} \middle| z \in \mathbb{R} \right\}.$

Let $z = 1$, $v_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$. #

$\lambda = 4$: $A - 4I = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$

$v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$

$\lambda = 2$: $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. #

Fact 3: Matrix A is singular if and only if A has a zero eigenvalue.

[To see this:]

A is singular $(\det A = 0) \Longleftrightarrow \ker A \neq \{0\}$

$\Longleftrightarrow 0 \neq v \in \ker A$

$\Longleftrightarrow Av = 0 = 0v$

$\Longleftrightarrow$ 0 is eigenvalue

v is eigenvector.

Fact 4: A and A^T have the same eigenvalues.

**However, the eigenvectors do not need to be the same.*

[To see this:] Characteristic equation for A:

$$0 = \det(A - \lambda I)$$

$$= \det(A - \lambda I)^T \quad \text{since } \det B = \det B^T.$$

$$= \underline{\det(A^T - \lambda I)}$$

characteristic equ. for A^T.

§ Basic Properties of Eigenvalues.

Let A is a $n \times n$ square matrix. Recall that its **characteristic polynomial**

$$\boxed{p_A(\lambda) = \det(A - \lambda I) = c_n\lambda^n + \cdots + c_1\lambda + c_0}$$

is a degree n polynomial, whose roots are the **eigenvalues** of A.

We can in principle factor p_A in the form

$$p_A(\lambda) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda).$$

We say that the eigenvalue λ_j has **multiplicity** k if it appears k times in the factorization of $p_A(\lambda)$.

Let's observe a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with eigenvalues λ_1, λ_2. Then

det $(A - \lambda I)$

$$p_A(\lambda) = det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = \lambda^2 - \underbrace{(a+d)}_{trA}\lambda + \underbrace{(ad - bc)}_{detA}$$

the same.

$$p_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) = \boxed{\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2}$$

Fact 5: The sum of all the eigenvalues equals the trace of $A = (a_{ij})_{n\times n}$:

$$\boxed{\text{tr}A = \sum_{i=1}^{n} a_{ii} = \lambda_1 + \cdots + \lambda_n.}$$

$\text{tr}A = a_{11} + \cdots + a_{nn}$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Furthermore, the product of all the eigenvalues equals the determinant of A:

$$\boxed{\det A = \prod_{i=1}^{n} \lambda_i = \lambda_1\lambda_2 \cdots \lambda_n.}$$

Revisit **Example 2.** Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & -1 & 5 \end{pmatrix}$. Check the following properties:

$\lambda = 6, 4, 2$

(1) $\mathsf{tr}A = \sum_{i=1}^{n} a_{ii} = \underbrace{\sum_{i=1}^{n} \lambda_i}$.

$2+5+5=12$ $\quad = 6+4+2=12.$

the same

(2) $\mathsf{det}A = \lambda_1\lambda_2\cdots\lambda_n$.

$\lambda_1\lambda_2\lambda_3 = 6\cdot 4\cdot 2 = 6\cdot 8 = 48$

det A: $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & -1 & 5 \end{pmatrix} \xrightarrow{(3)+\frac{1}{5}(2)} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & 0 & 5+(-\frac{1}{5}) \end{pmatrix}$

det $A = 2\cdot 5\cdot(5-\frac{1}{5})$

$= 48$

Example 3. Let $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Check (1) $\mathsf{tr}A = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$.

(2) $\mathsf{det}A = \lambda_1\lambda_2\cdots\lambda_n$.

(1) $\det(A-\lambda I) = 0$.

$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 & 0 \\ 1 & 3-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}$.

$= (2-\lambda)\det\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$.

$\lambda = \underline{2}, \underline{2}, \underline{\underline{4}}$.

$\lambda=2$, multiplicity $\underline{\underline{2}}$.

[To be continued!]