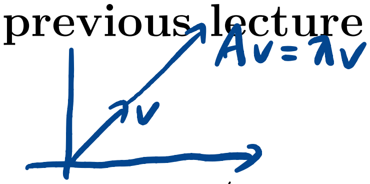
Lecture 31: Quick review from previous lecture



Let $A = (a_{ij})$ be $n \times n$ matrix.

- If λ is an **eigenvalue** of A, then there exists a vector $\mathbf{v} \neq \mathbf{0}$ satisfying $A\mathbf{v} = \lambda\mathbf{v}$. We call $\mathbf{v}$ is an **eigenvector**.
- λ is an eigenvalue of the matrix A if and only if λ is a solution to the **characteristic equation**

$$\det(A - \lambda I) = 0.$$

- A and A^T ~~has~~ have the same eigenvalues.
- $\underline{\text{tr}A} = \sum_{i=1}^n a_{ii} = \underline{\lambda_1 + \cdots + \lambda_n}$.
- $\text{det}A = \lambda_1\lambda_2 \cdots \lambda_n$.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

① $\text{Tr}\, A = a_{11} + \cdots + a_{nn}$
$= \lambda_1 + \cdots + \lambda_n$

② $\det A = \lambda_1 \cdots \lambda_n$.

Today we will discuss

- Section 8.3 Eigenvector Bases.

- Lecture will be recorded -

Example 3. Let $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Check (1) $\text{tr}A = \sum_{i=1}^{n} \lambda_i$.

(2) $\det A = \lambda_1 \lambda_2 \cdots \lambda_n$.

Find eigenvalues : Set up $\det(A-\lambda I)=0$.

$$0 = \det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 1 & 0 \\ 1 & 3-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

$$= (2-\lambda)\det\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$$

$$= (2-\lambda)[(3-\lambda)^2 - 1]$$

$$= (2-\lambda)(\lambda^2 - 6\lambda + \boxed{8})$$

$$= (2-\lambda)(\lambda-2)(\lambda-4)$$

1 -2
8 -4

Then $\lambda = 2, 2, 4$

「 $\lambda = 2$ has multiplicity 2.

$\lambda = 4$ 〃 1 」

(1) $\text{Tr}\, A = 3 + 3 + 2 = 8$. the same

$\lambda_1 + \lambda_2 + \lambda_3 = 2 + 2 + 4 = 8$

(2) $\det A = \det\begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$= 2\det\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = 2 \cdot 8 = 16$$

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 2 \cdot 2 \cdot 4 = 16$$

Then $\det A = \lambda_1 \lambda_2 \lambda_3$

Find eigenvectors: $\ker(A - \lambda I)$.

$\lambda = 2$: $A - 2I = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{②-①} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$\ker(A - 2I) = \left\{ \begin{pmatrix} -y \\ y \\ z \end{pmatrix} \;\middle|\; y, z \in \mathbb{R} \right\}$

$y = 1, z = 0$ $\qquad$ $z = 1, y = 0$

$v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\qquad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$\lambda = 4$:

$v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. #

§ Similar matrices

Definition: Let A and B be $n \times n$ square matrices. We say that B is **similar** to A if there exists an invertible matrix S so that

$$B = S^{-1}AS.$$

We actually saw this before.

Recall that a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$, $L[\mathbf{x}] = A\mathbf{x}$. The **matrix representation** of L in the basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is

$$B = S^{-1}AS, \qquad \text{where } S = [\mathbf{v}_1, \ldots, \mathbf{v}_n].$$

Fact 6: Let A and B be $n \times n$ square matrices. If B is **similar** to A, then

det $(A - \lambda I)$, det $(B - \lambda I)$

(1) A and B share the same characteristic polynomial and eigenvalues.

(2) Moreover, if $B\mathbf{w} = \mu\mathbf{w}$ and μ is then eigenvalue of B, then $S\mathbf{w}$ is an eigenvector of A with eigenvalue μ.
Similarly, if $A\mathbf{v} = \lambda\mathbf{v}$ and λ is the eigenvalue of A, then $S^{-1}\mathbf{v}$ is an eigenvector of B with eigenvalue λ.

$S^{-1}(\lambda I)S = \lambda(S^{-1}IS)$
$= \lambda(S^{-1}S)$
$= \lambda I$

[To see this:]

$$(1)\ 0 = \det(B - \lambda I) = \det(S^{-1}AS - \underbrace{\lambda I}_{S^{-1}(\lambda I)S})$$
$$= \det(S^{-1}(A - \lambda I)S)$$
$$= \cancel{\det S^{-1}}\det(A - \lambda I)\cancel{\det S}$$
$$= \det(A - \lambda I)$$

This implies A, B have the same eigenvalues

(2) $Bw = \mu w$

$B = S^{-1}AS$ $\Rightarrow (S^{-1}AS)w = \mu w$

multiply S $\Rightarrow A\,Sw = \mu(Sw)$. Thus Sw is an eigenvector of A.

.

For example,

Example 4 . Let's consider

$$\underbrace{\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}}_{B} = S^{-1}AS, \qquad \text{where } A = \begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

It is clear that B is similar to A.

The matrix B has eigenvalues 3 and 2, with eigenvectors

$$\mathbf{w}_1 = (1,1)^T, \qquad \mathbf{w}_2 = (2,1)^T \quad \text{(check this!)}.$$

Then **Fact 6** implies that A also has eigenvalues 3 and 2, but has eigenvectors

$$S\mathbf{w}_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad S\mathbf{w}_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

eigenvectors of A.

8.3 Eigenvector Bases

Fact 1: Eigenvectors corresponding to different eigenvalues are linearly independent.

More generally, if $\lambda_1, \ldots, \lambda_k$ are pairwise distinct eigenvalues of A, then the corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

$\lambda_1 \neq \lambda_2$, $Av_1 = \lambda_1 v_1$ ①

$Av_2 = \lambda_2 v_2$ ②

Claim: v_1, v_2 are l. indep.

By contradiction argument, suppose v_1, v_2 are l. dep.

$(v_1 = c v_2,\ c \neq 0)$.

$$\underset{\lambda_1 v_1}{\overset{①}{Av_1}} \overset{v_1 = cv_2}{=} A(cv_2) = cAv_2 \overset{②}{=} c\lambda_2 v_2 \overset{v_1 = cv_2}{=} \lambda_2 v_1$$

Then $\lambda_1 = \lambda_2$. (impossible!)

So, v_1, v_2 are l. indep #

Thus, from the above **Fact 1**, we can derive that

Fact 2: If $n \times n$ real matrix A has n distinct real eigenvalues $\lambda_1, \cdots, \lambda_n$, then their corresponding real eigenvectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ form a basis of $\mathbb{R}^n$.

(Fact 1 gives $v_1, \ldots, v_n$ are l. indep.)

Definition: We say that an eigenvalue λ of a matrix A is **complete** if the number of linearly independent eigenvectors with eigenvalue λ is equal to the multiplicity of λ.

Definition: If A is a matrix with eigenvalue λ, we define the eigenspace of λ to be
$$V_\lambda = \ker(A - \lambda I).$$

Then

$$\begin{aligned} &\dim V_\lambda \quad \text{(number of free variables)} \\ = &\text{number of linearly indept. eigenvectors of } A \text{ with eigenvalue } \lambda. \end{aligned}$$

If $\boxed{\dim V_\lambda = \text{the multiplicity of } \lambda}$, then λ is complete.

Example 1.

(1) $A = \begin{pmatrix} c & 1 \\ 0 & c \end{pmatrix}$ $0 = \det(A - \lambda I) = \det\begin{pmatrix} c-\lambda & 1 \\ 0 & c-\lambda \end{pmatrix} = (c-\lambda)^2.$

$\lambda = c, c$ (multiplicity 2).

$V_\lambda = \ker(A - cI) = \ker\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \middle| \ x \in \mathbb{R} \right\}$

$\dim V_c = 1$. Then $\lambda = c$ is <u>NOT</u> complete.

(2) On the other hand, we consider $B = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$

$\lambda = c, c$.

$\ker(B - cI) = \ker\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \ x, y \in \mathbb{R} \right\}$.

$\dim V_c = 2$. Then $\lambda = c$ is complete.

B is complete

Definition: If all eigenvalues of A are complete, we say the matrix A itself is a **complete matrix**.

§ Diagonalization.

Goal: $$A \longrightarrow \begin{pmatrix} * & 0 & \dots & 0 \\ 0 & * & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & * \end{pmatrix}$$

Consider the linear operator $\underline{L[\mathbf{v}] = A\mathbf{v}}$. Suppose matrix A is complete. Then A has eigenvalues $\lambda_1, \dots, \lambda_n$ and their corresponding eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ forms a basis of $\mathbb{R}^n$

$Av_j = \lambda_j v_j \ , \ 1 \le j \le n.$

$$A \underbrace{[v_1 \ \cdots \ v_n]}_{V} = [\lambda_1 v_1 \ \cdots \ \lambda_n v_n] = \underbrace{[v_1 \cdots v_n]}_{V} \overbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}^{D}$$

$$AV = V \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \Rightarrow A = VDV^{-1} . \ \#$$

From above, we have shown that we can factor any complete matrix A:

Fact 3: Suppose matrix A is complete. Then

$$\boxed{A = VDV^{-1}}$$

where $V = [\mathbf{v}_1, \dots, \mathbf{v}_n]$ is the matrix of eigenvectors, and

$$D = \operatorname{diag}(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}.$$

✓ In other words, matrix representation of the operator $L[\mathbf{v}] = A\mathbf{v}$ in the basis of A's eigenvectors gives a diagonal matrix D. (that is, $D = V^{-1}AV$.)

Definition: We say that the matrix A is **diagonalizable**, meaning it can be factored in the form

$$A = VDV^{-1}, \quad \text{where } D \text{ is diagonal and } V \text{ is nonsingular.}$$