Lecture 32: Quick review from previous lecture

- Let A and B be $n \times n$ square matrices. We say that B is **similar** to A if there exists an invertible matrix S so that

$$B = S^{-1} A S.$$

① B, A have the same eigenvalue

- Eigenvectors corresponding to different eigenvalues are linearly independent.
- If $\boxed{\dim \ker(A - \lambda I) = \text{the multiplicity of } \lambda}$, then λ is complete.

numbers of free variables

- If all eigenvalues of A are complete, we say the matrix A itself is a **complete matrix**.

Today we will

- continue Section 8.3 Eigenvector Bases.

- Lecture will be recorded -

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§ Diagonalization.

Goal: $$A \longrightarrow \begin{pmatrix} * & 0 & \dots & 0 \\ 0 & * & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & * \end{pmatrix}$$

Consider the linear operator $L[\mathbf{v}] = A\mathbf{v}$. Suppose matrix A is complete. Then A has eigenvalues $\lambda_1, \dots, \lambda_n$ and their corresponding eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ forms a basis of $\mathbb{R}^n$.

basis of $\mathbb{R}^n$.

$A v_j = \lambda_j v_j, \ 1 \le j \le n$

$$A \underbrace{[v_1 \ \cdots \ v_n]}_{V} = [\lambda_1 v_1 \ \cdots \ \lambda_n v_n] = \underbrace{[v_1 \ \cdots \ v_n]}_{V} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{D}$$

$$A = V D V^{-1}.$$

From above, we have shown that we can factor any complete matrix A:

Fact 3: Suppose matrix A is complete. Then

$$\boxed{A = VDV^{-1}}$$

where $V = [\mathbf{v}_1, \dots, \mathbf{v}_n]$ is the matrix of eigenvectors, and

$$D = \operatorname{diag}(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}.$$

✓ In other words, matrix representation of the operator $L[\mathbf{v}] = A\mathbf{v}$ in the basis of A's eigenvectors gives a diagonal matrix D. (that is, $D = V^{-1}AV$.)

matrix representation of L in basis $\{v_1, \dots, v_n\}$.

Definition: We say that the matrix A is **diagonalizable**, meaning it can be factored in the form

$$A = VDV^{-1}, \quad \text{where } D \text{ is diagonal and } V \text{ is nonsingular.}$$

Fact 4: A matrix is complete if and only if it is diagonalizable.

Let's revisit Examples 3 in Section 8.2.

Example 2. Diagonalize the matrix $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

[Answer:] We have found its eigenvalues $\lambda = \underbrace{2, 2}_{\text{multiplicity } 2.}, 4$.

(1) Eigenvalue $\lambda = 2$: The eigenvectors are

$$\mathbf{v}_1 = (-1, 1, 0)^T, \qquad \mathbf{v}_2 = (0, 0, 1)^T.$$

(2) Eigenvalue $\lambda = 4$: The eigenvector is

$$\mathbf{v}_3 = (1, 1, 0)^T.$$

Thus, the matrix A is complete. Then we can factor A into

$$A = V D V^{-1}, \text{ where } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \ V = [v_1 \ v_2 \ v_3].$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \underbrace{V^{-1}}$$

exercise: find V^{-1}

by using Gauss – Jordan

§ Some properties.

> **Q:** Suppose $A = VDV^{-1}$, where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ (that is, A is diagonalizable.) What is A^2 and A^k?

(1) $A^2 = (VD\underbrace{V^{-1})(V}_{I}DV^{-1}) = VDDV^{-1} = VD^2V^{-1}$

$= V\begin{bmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{bmatrix} V^{-1}$

eigenvalues of A^2: $\lambda_1^2, \ldots, \lambda_n^2$;

eigenvectors of A^2: same as A.

(2) $A^k = \overbrace{(VDV^{-1}) \cdots (VDV^{-1})}^{k} = VD^kV^{-1}$

Thus, we conclude

> **Fact 5:** A^k has the same eigenvectors as A, and the eigenvalues are just $\lambda_1^k, \ldots, \lambda_n^k$.

Example 3: Once again, we consider the matrix $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Find A^k for any positive integer k.

$$A = \underbrace{\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}}_{D} \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1}$$

$$A^k = V D^k V^{-1}$$

$$= V \begin{pmatrix} 2^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 4^k \end{pmatrix} V^{-1}. \quad \#$$

Recall that if B is **similar** to A, then A and B share the eigenvalues.

Definition: We say two matrices A and B are **simultaneously diagonalizable** if

$$A = VD_1V^{-1} \text{ and } B = VD_2V^{-1}, \qquad \text{diagonal matrices } D_1 \text{ and } D_2.$$

$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ $\begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix}$

Fact 6: If A and B are **simultaneously diagonalizable**, then the product AB has the same eigenvectors as A and B. Moreover, the eigenvalues of AB are just the products of the eigenvalues of A and B.

[To see this:]

$$AB = V D_1 \underbrace{V^{-1} V}_{I} D_2 V^{-1} = V D_1 D_2 V^{-1}$$

$\boxed{BA}$

$$= V \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix} V^{-1}$$

$$= V \begin{bmatrix} \lambda_1\mu_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n\mu_n \end{bmatrix} V^{-1}.$$

Thus, AB has eigenvalues $\lambda_j \mu_j$

the same eigenvectors as A, B.

Q: If A, B are simultaneously diagonalizable, then $AB = BA$ (Yes)

Fact 7: If $A = VDV^{-1}$, then A is invertible if and only if $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ has all nonzero diagonal elements.

Moreover, A^{-1} has eigenvalues $1/\lambda_i$, $1 \le i \le n$.

[To see this:]

(1) $0 \ne \det A = \det(VDV^{-1}) = \cancel{\det V} \det D \cancel{\det V^{-1}}$

$= \det D$

$= \lambda_1 \cdots \lambda_n$. #

(2) $A^{-1} = (VDV^{-1})^{-1} = (V^{-1})^{-1} D^{-1} V^{-1}$

$= V \begin{bmatrix} 1/\lambda_1 & & 0 \\ & \ddots & \\ 0 & & 1/\lambda_n \end{bmatrix} V^{-1}$. #

§ Systems of Differential Equations.

Example 4. Find the **general solutions** to this system of differential equations:

$$\begin{aligned} x_1' &= 3x_1 + x_2 + x_3 \\ x_2' &= 2x_1 + 4x_2 + 2x_3 \\ x_3' &= -x_1 - x_2 + x_3, \end{aligned} \Rightarrow \underbrace{\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}}_{\vec{x}'(t)} = \underbrace{\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\vec{x}(t)}$$

where $x_i = x_i(t)$ is a differentiable real-valued function of the real variable t.

* Clearly, $x_i(t) = 0$ is the solution of the system.

Solve $\vec{x}'(t) = A\vec{x}$. Diagonalize A:

$$A = V\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}V^{-1}, \quad V = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix}$$

exercise!

$$\vec{x}'(t) = VDV^{-1}\vec{x}(t)$$

multiply V^{-1}

$$\Rightarrow V^{-1}\vec{x}' = DV^{-1}\vec{x}$$

Let $\vec{y} = V^{-1}\vec{x}$

$$\Rightarrow \boxed{\vec{y}' = D\vec{y}}$$

New system

$$\begin{aligned} y_1' &= 2y_1 & &\Rightarrow & y_1(t) &= c_1e^{2t} \\ y_2' &= 2y_2 & &\Rightarrow & y_2(t) &= c_2e^{2t} \\ y_3' &= 4y_3 & &\Rightarrow & y_3(t) &= c_3e^{4t} \end{aligned}$$

Back to $\vec{x} = V\vec{y}$

$$= \begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$$
$$= \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1e^{2t} \\ c_2e^{2t} \\ c_3e^{4t} \end{pmatrix}$$

$$= \begin{pmatrix} -c_1e^{2t} - c_3e^{4t} \\ -c_2e^{2t} - 2c_3e^{4t} \\ c_1e^{2t} + c_2e^{2t} + c_3e^{4t} \end{pmatrix} = e^{2t}\overbrace{(c_1v_1 + c_2v_2)}^{\ker(A-2I)} + e^{4t}(\underbrace{c_3v_3}_{\ker(A-4I)})$$

Poll Question 1: Can the eigenvalue of a square matrix be zero?

A) Yes

B) No