Lecture 33: Quick review from previous lecture

- The matrix A is **diagonalizable** if it can be factored in the form

$$A = VDV^{-1} = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}^{-1}$$

  where D is diagonal and V is nonsingular.

  λ_j : eigenvalues,
  u_j : eigenvectors.

- A matrix is complete if and only if it is diagonalizable.

$$= \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} \ddots & 0 \\ 0 & \end{bmatrix} \begin{bmatrix} & \end{bmatrix}^{-1}$$

Today we will

- Section 8.5 Eigenvalues of Symmetric Matrices.

- Lecture will be recorded -

- HW 11 due Today at 6pm.

8.5 Eigenvalues of Symmetric Matrices

Let's focus on the theory of eigenvalues and eigenvectors for **symmetric matrices**, which have many nice properties.

Recall the example again.

Example 1. $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. In Lecture 31, we have found

$\langle v_1, v_2 \rangle = 0$ $\quad \langle v_2, v_3 \rangle = 0$
$\langle v_1, v_3 \rangle = 0$

eigenvalue $\lambda = 2, 2$ eigenvectors $\mathbf{v}_1 = (-1, 1, 0)^T$, $\mathbf{v}_2 = (0, 0, 1)^T$,
eigenvalue $\lambda = 4$, eigenvector $\mathbf{v}_3 = (1, 1, 0)^T$.

Thus, the matrix A is complete. Moreover,

$$A = VDV^{-1},$$

where $D = \text{diag}(2, 2, 4)$ and $V = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$.

- These eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are mutually orthogonal!
- The eigenvalues of A are real numbers, not complex.

Recall: $A v_1 = \lambda_1 v_1$, $A v_2 = \lambda_2 v_2$, $\lambda_1 \neq \lambda_2 \xRightarrow{\text{Previously}} \{v_1, v_2\}$ l. indep.

Now if $A = A^T$, then $\{v_1, v_2\}$ are orthogonal.

These facts are explained by the following Theorem.

Fact 1: Let $A = A^T$ be a real symmetric $n \times n$ matrix. Then

1. All the eigenvalues of A are real.
2. Eigenvectors corresponding to **distinct** eigenvalues are orthogonal.
3. There is an orthonormal basis of $\mathbb{R}^n$ consisting of n eigenvectors of A.

In particular, all real symmetric matrices are complete and real diagonalizable.

* Orthogonality is with respect to the standard dot product on $\mathbb{R}^n$.

Its proof can be found in the textbook.

Suppose A is **real and symmetric**, and let $\lambda_1, \ldots, \lambda_n$ denote its eigenvalues. Then the above Theorem tells us we can choose eigenvectors

$$\mathbf{u}_1, \ldots, \mathbf{u}_n \quad (\text{so } A\mathbf{u}_i = \lambda_i \mathbf{u}_i) \text{ that are } \textbf{orthonormal}.$$

If $\boxed{Q = [\mathbf{u}_1, \ldots, \mathbf{u}_n]}$ and $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, then

$\left\{ \begin{array}{l} Q \text{ is orthogonal matrix.} \\ Q^TQ = I = QQ^T \Rightarrow Q^{-1} = Q^T \end{array} \right.$

$A [u_1 \cdots u_n] = [Au_1 \cdots Au_n] = [\lambda_1 u_1 \cdots \lambda_n u_n]$

$= \underbrace{[u_1 \cdots u_n]}_{Q} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{D}$

$A = Q D \boxed{Q^{-1}} = Q D Q^T . = [\text{orthogonal}] \begin{bmatrix} \diagdown & 0 \\ 0 & \end{bmatrix} [\text{orthogonal}]^{-1}$

Thus, we conclude that

Fact 2: (The Spectral Theorem) Let $A = A^T$ be a real symmetric $n \times n$ matrix. Then there exists an orthogonal matrix Q such that

$$A = QDQ^{-1} = QDQ^T, \qquad (\text{spectral factorization})$$

where D is a real diagonal matrix. The eigenvalues of A appear on the diagonal of D, while the columns of Q are the corresponding orthonormal eigenvectors.

* The term "spectrum" refers to the eigenvalues of a matrix.

Example 2. Find the spectral factorization of $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

[Answer:] From **Example 1**, we have seen $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ has "orthogonal" eigenvectors

$\lambda = 2 \qquad 2 \qquad 4$

$$\mathbf{v}_1 = (-1, 1, 0)^T, \quad \mathbf{v}_2 = (0, 0, 1)^T, \quad \mathbf{v}_3 = (1, 1, 0)^T.$$

[Previously, diagonalize A: $A = [v_1\ v_2\ v_3] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} [v_1\ v_2\ v_3]^{-1}$]

Normalize $v_1 \quad v_2 \quad v_3$:

$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad q_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$

Spectral factorization:

$$A = \underbrace{\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}}_{Q} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^T.$$

Q is orthogonal matrix $(Q^T = Q^{-1})$.

Recall:

We denote $K > 0$. if K is positive definite.

① $K = K^T$

② $x^T K x > 0$ if $x \neq 0$

§ **Revisit Positive definite matrix.** Suppose K is positive definite (in particular, symmetric). Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ denote the orthonormal eigenvector basis, with eigenvalues $\lambda_1, \ldots, \lambda_n$ for matrix K. (By Fact 2), $K u_j = \lambda_j u_j$.

> **Fact 3:** A symmetric matrix K is positive definite if and only if all of its eigenvalues are strictly positive, that is, $\lambda_j > 0$

[To see this:]

($\Rightarrow$) Since $K > 0$, $x^T K x > 0$ if $x \neq 0$, $x \in \mathbb{R}^n$.

$$0 < u_j^T K u_j = u_j^T \lambda_j u_j = \lambda_j (u_j^T u_j).$$

$$= \lambda_j . \quad \text{since } \|u_j\|^2 = 1$$

$$\|u_j\|^2 = u_j^T u_j$$

($\Leftarrow$) Claim: $x^T K x > 0$ for all $x \neq 0$, $x \in \mathbb{R}^n$

Since $\{u_1, \ldots, u_n\}$ is O.N.B, we can write

$$x = c_1 u_1 + \cdots + c_n u_n.$$

$$x^T K x = (c_1 u_1 + \cdots + c_n u_n)^T K (c_1 u_1 + \cdots + c_n u_n)$$

$$= (c_1 u_1 + \cdots + c_n u_n)^T (c_1 \lambda_1 u_1 + \cdots + c_n \lambda_n u_n)$$

$$= c_1^2 \lambda_1 u_1^T u_1 + \cdots + c_n^2 \lambda_n u_n^T u_n$$

$\{u_j\}$ O.N.B.

① $\|u_j\| = 1$

② $\langle u_j, u_i \rangle = 0$ $i \neq j$

$$= c_1^2 \lambda_1 + \cdots + c_n^2 \lambda_n$$

> 0 since $\lambda_j > 0$, $c_1, \ldots, c_n$ are not all zero. #

$\Rightarrow K > 0$. #

Remark: The same proof shows that K is positive semidefinite if and only if all its eigenvalues $\lambda \geq 0$.

$K \geq 0 \iff \lambda \geq 0$.

$$\begin{cases} x^T K x \geq 0 \\ K^T = K \end{cases}$$

Example 3. Determine if $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & -1 & 5 \end{pmatrix}$ is positive definite.

[Answer:]

1. Method 1: In Lec.30, A has eigenvalues $2, 4, 6$.

By Fact 3, since $A^T = A$ has positive eigenvalue, we have $A > 0$. #

2. Method 2: **To see if a matrix is positive definite**, one can also perform the Gaussian elimination (See in Lecture 19):

Proof. From Gaussian elimination, we have

pivots

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & -1 & 5 \end{pmatrix} - - - - - \longrightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & 0 & 24/5 \end{pmatrix}.$$

Since all diagonal entries are positive, we confirm that A is positive definite. □

Fact 4: If $A = A^T$ is symmetric, suppose $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are the orthonormal eigenvectors. Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_r$ all have non-zero eigenvalues, but $\mathbf{u}_{r+1}, \ldots, \mathbf{u}_n$ have eigenvalue 0 (i.e. they're in $\ker A$). Consequently, $\mathbf{u}_1, \ldots, \mathbf{u}_r$ are orthogonal to $\ker A$, and hence

$$\mathbf{u}_1, \ldots, \mathbf{u}_r \text{ form an orthonormal basis for } \operatorname{coimg} A = \operatorname{img} A.$$

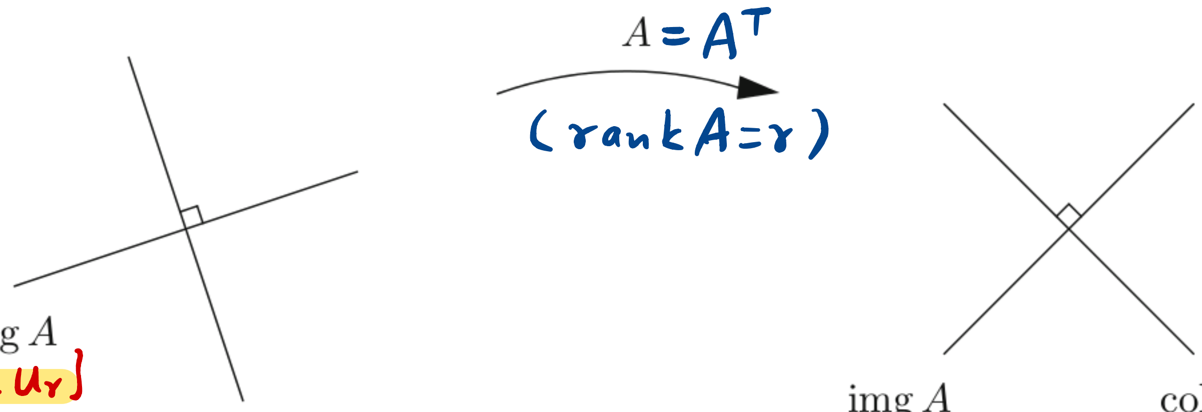
Moreover, one has

$$\mathbf{u}_{r+1}, \ldots, \mathbf{u}_n \text{ form an orthonormal basis for } \ker A = \operatorname{coker} A.$$

$\lambda_1^{\neq 0}, \ldots, \lambda_r^{\neq 0} \longrightarrow \{u_1, \ldots, u_r\}$ $\operatorname{img} A = \operatorname{coimg} A$

$Au_{r+1} = 0u_{r+1}$ $\lambda_{r+1}^{=0}, \ldots, \lambda_n^{=0} \longrightarrow$

$= 0, \Rightarrow u_{r+1} \in \ker A$



Example 4. Let $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Use Fact 4 to find an orthonormal basis for coimg A and img A.

$\Rightarrow A = A^T$.

O.N.B.

① Find eigenvalues: $0 = \det(A - \lambda I) \Rightarrow \lambda = 3, 2, 0$.

② Find eigenvectors:

$\lambda = 3$: $A - 3I = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. $\ker(A - 3I)$ has a basis $v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 2$: $A - 2I = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $\ker(A - 2I)$ has a basis $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\lambda = 0$: $v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

$\{v_1, v_2, v_3\}$ are orthogonal, but NOT yet orthonormal

normalize $\Longrightarrow$ $q_j = \cancel{\dfrac{v_j}{\|v_j\|}}$, $\underset{q_1}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$, $\underset{q_2}{\dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}$, $\underset{q_3}{\dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$

Fact 4 $\Rightarrow$ $\{q_1, q_2\}$ is O.N.B for img A, coimg A.

spectral factorization: $A = [q_1\ q_2\ q_3] \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} [q_1\ q_2\ q_3]^T$