Lecture 33: Quick review from previous lecture

- The matrix $A$ is **diagonalizable** if it can be factored in the form
  \[ A = V D V^{-1} = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}^\top \]

  where $D$ is **diagonal** and $V$ is **nonsingular**.

- A matrix is **complete** if and only if it is diagonalizable.

Today we will

- Section 8.5 Eigenvalues of **Symmetric** Matrices.

  - Lecture will be recorded -

- HW 11 due Today at 6pm.
8.5 Eigenvalues of Symmetric Matrices

Let’s focus on the theory of eigenvalues and eigenvectors for symmetric matrices, which have many nice properties.

Recall the example again.

**Example 1.** \( A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \). In Lecture 31, we have found

\[
\begin{align*}
\langle v_1, v_2 \rangle &= 0 \\
\langle v_2, v_3 \rangle &= 0 \\
\langle v_1, v_3 \rangle &= 0
\end{align*}
\]

eigenvalue \( \lambda = 2, 2 \) \ eigenvectors \( v_1 = (-1, 1, 0)^T \), \( v_2 = (0, 0, 1)^T \),
eigenvalue \( \lambda = 4 \) \ eigenvector \( v_3 = (1, 1, 0)^T \).

Thus, the matrix \( A \) is complete. Moreover,

\[
A = VDV^{-1},
\]

where \( D = \text{diag}(2, 2, 4) \) and \( V = [v_1, v_2, v_3] \).

- These eigenvectors \( v_1, v_2, v_3 \) are **mutually orthogonal**!
- The eigenvalues of \( A \) are **real numbers**, not complex.

**Recall:** \( Av_1 = \lambda_1 v_1 \), \( Av_2 = \lambda_2 v_2 \) \( \implies \langle v_1, v_2 \rangle \perp \) \( \text{indep} \).

Now if \( A = A^T \) then \( \langle v_1, v_2 \rangle \perp \) \( v_1, v_2 \) are **orthogonal**.

These facts are explained by the following Theorem.

**Fact 1:** Let \( A = A^T \) be a **real symmetric** \( n \times n \) matrix. Then

1. All the eigenvalues of \( A \) are **real**.
2. Eigenvectors corresponding to **distinct** eigenvalues are **orthogonal**.
3. There is an **orthonormal basis** of \( \mathbb{R}^n \) consisting of \( n \) eigenvectors of \( A \).

In particular, all **real symmetric** matrices are **complete** and **real diagonalizable**.

* Orthogonality is with respect to the standard dot product on \( \mathbb{R}^n \).
  Its proof can be found in the textbook.
Suppose $A$ is **real and symmetric**, and let $\lambda_1, \ldots, \lambda_n$ denote its eigenvalues. Then the above Theorem tells us we can choose eigenvectors

$$u_1, \ldots, u_n$$

(so $Au_i = \lambda_i u_i$) that are **orthonormal**.

If $Q = [u_1, \ldots, u_n]$ and $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$, then

$$A [u_1 \ldots u_n] = [Au_1 \ldots Au_n] = [\lambda_1 u_1 \ldots \lambda_n u_n]$$

$$= [u_1 \ldots u_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

Thus, we conclude that

$$A = QDQ^{-1} = QDQ^T.$$  

**Fact 2: (The Spectral Theorem)** Let $A = A^T$ be a real symmetric $n \times n$ matrix. Then there exists an orthogonal matrix $Q$ such that

$$A = QDQ^{-1} = QDQ^T, \quad \text{(spectral factorization)}$$

where $D$ is a real diagonal matrix. The eigenvalues of $A$ appear on the diagonal of $D$, while the columns of $Q$ are the corresponding orthonormal eigenvectors.

* The term “spectrum” refers to the eigenvalues of a matrix.

**Example 2.** Find the **spectral factorization** of $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

[Answer:] From **Example 1**, we have seen $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ has “orthogonal” eigenvectors

$$v_1 = (-1, 1, 0)^T, \; v_2 = (0, 0, 1)^T, \; v_3 = (1, 1, 0)^T.$$

$$\text{diagonalize } A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T.$$

**Normalize** $v_1, v_2, v_3$:

$$v_1 = \frac{v_1}{||v_1||} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \; v_2 = (0, 0, 1)^T, \; v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$
Spectral factorization:

$$A = \begin{bmatrix} 8_1 & 9_2 & 9_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 9_1 & 9_2 & 9_3 \end{bmatrix}^T$$

$Q$ is orthogonal matrix ($Q^T = Q^{-1}$).

Recall:

We denote $K > 0$ if $K$ is positive definite.

1. $K = K^T$
2. $x^T K x > 0$ if $x \neq 0$
§ Revisit Positive definite matrix. Suppose $K$ is positive definite (in particular, symmetric). Let $u_1, \ldots, u_n$ denote the orthonormal eigenvector basis, with eigenvalues $\lambda_1, \ldots, \lambda_n$ for matrix $K$. (By Fact 2) $K u_j = \lambda_j u_j$.

**Fact 3:** A symmetric matrix $K$ is positive definite if and only if all of its eigenvalues are strictly positive, that is, $\lambda_j > 0$

[To see this:]

$(\Rightarrow)$ Since $K > 0$, $x^T K x > 0$ if $x \neq 0, x \in \mathbb{R}^n$.

$$0 < u_j^T K u_j = u_j^T \lambda_j u_j = \lambda_j (u_j^T u_j)$$

$$= \lambda_j \cdot \text{since } \|u_j\|^2 = 1$$

$(\Leftarrow)$ Claim: $x^T K x > 0$ for all $x \in \mathbb{R}^n$.

Since $\{u_1, \ldots, u_n\}$ is O.N.B, we can write

$$x = c_1 u_1 + \cdots + c_n u_n.$$ 

$$x^T K x = (c_1 u_1 + \cdots + c_n u_n)^T K (c_1 u_1 + \cdots + c_n u_n)$$

\[ \{ u_j \} \text{ O.N.B.} \]

1. $\|u_j\| = 1$
2. $\langle u_j, u_i \rangle = 0$ for $j \neq i$.

$\Rightarrow$ since $\lambda_j > 0$, $c_1, \ldots, c_n$ are not all zero. 

Remark: The same proof shows that $K$ is positive semidefinite if and only if all its eigenvalues $\lambda \geq 0$.

\[ \{ x^T K x \geq 0 \} \text{ if and only if } \lambda \geq 0. \]
Example 3. Determine if $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & -1 & 5 \end{pmatrix}$ is positive definite.

[Answer:]

1. Method 1: In Lec. 30, $A$ has eigenvalues 2, 4, 6. By Fact 3, since $A^T = A$ has positive eigenvalue, we have $A > 0$. 

2. Method 2: To see if a matrix is positive definite, one can also perform the Gaussian elimination (See in Lecture 19):

Proof. From Gaussian elimination, we have

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & -1 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & 0 & 24/5 \end{pmatrix}. $$

Since all diagonal entries are positive, we confirm that $A$ is positive definite.

Fact 4: If $A = A^T$ is symmetric, suppose $u_1, \ldots, u_n$ are the orthonormal eigenvectors. Suppose $u_1, \ldots, u_r$ all have non-zero eigenvalues, but $u_{r+1}, \ldots, u_n$ have eigenvalue 0 (i.e. they’re in ker $A$). Consequently, $u_1, \ldots, u_r$ are orthogonal to ker $A$, and hence

$u_1, \ldots, u_r$ form an orthonormal basis for coimg $A = \text{img} A$.

Moreover, one has

$u_{r+1}, \ldots, u_n$ form an orthonormal basis for ker $A = \text{coker} A$.
Example 4. Let \( A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \). Use Fact 4 to find an orthonormal basis for \( \text{coimg } A \) and \( \text{img } A \).

1. **Find eigenvalues:** \( 0 = \det(A - \lambda I) \Rightarrow \lambda = 3, 2, 0 \).

2. **Find eigenvectors:**
   - \( \lambda = 3 \): \( A - 3I = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \). \( \text{ker}(A - 3I) \) has a basis \( v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \).
   - \( \lambda = 2 \): \( A - 2I = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \). \( \text{ker}(A - 2I) \) has a basis \( v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \).
   - \( \lambda = 0 \): \( v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \).

\{ v_1, v_2, v_3 \} are orthogonal, but not yet orthonormal.

- Normalize \( g_i = \frac{v_i}{\|v_i\|} \) to get \( \{ g_1, g_2, g_3 \} \) is orthonormal.

\[ A^* = A^T \]

Fact 4: \( \{ g_1, g_2, g_3 \} \) is O.N.B. for \( \text{img } A \), \( \text{coimg } A \).

Spectral factorization: \( A = \begin{pmatrix} g_1 & g_2 & g_3 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} g_1 & g_2 & g_3 \end{pmatrix} \)