Lecture 34: Quick review from previous lecture

- $$A \text{ is complete} \Longrightarrow A = VDV^{-1} = V \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix} V^{-1}$$

  where D is diagonal and V is nonsingular.

- $$\begin{aligned} & A \text{ is a } n \times n \text{ real, symmetric matrix} \\ \Longrightarrow\ & \text{there exists orthonormal eigenvector basis } \mathbf{u}_1, \ldots, \mathbf{u}_n \\ & \text{to eigenvalues } \lambda_1, \ldots, \lambda_n \\ \Longrightarrow\ & A = QDQ^{-1} = QDQ^T, \ \text{(spectral factorization)} \\ & \text{where } D = \operatorname{diag}(\underline{\lambda_1, \ldots, \lambda_n}) \text{ and } \underline{Q = [\mathbf{u}_1 \ldots \mathbf{u}_n]}. \end{aligned}$$

  Q is orthogonal matrix
  $(Q^{-1} = Q^T)$

Today we will

- Continue Section 8.5 Eigenvalues of Symmetric Matrices.

- Lecture will be recorded -

---

§ Some Geometric Observation.

In $\mathbb{R}^2$, the spectral factorization of a **symmetric** matrix A has a natural geometric interpretation.

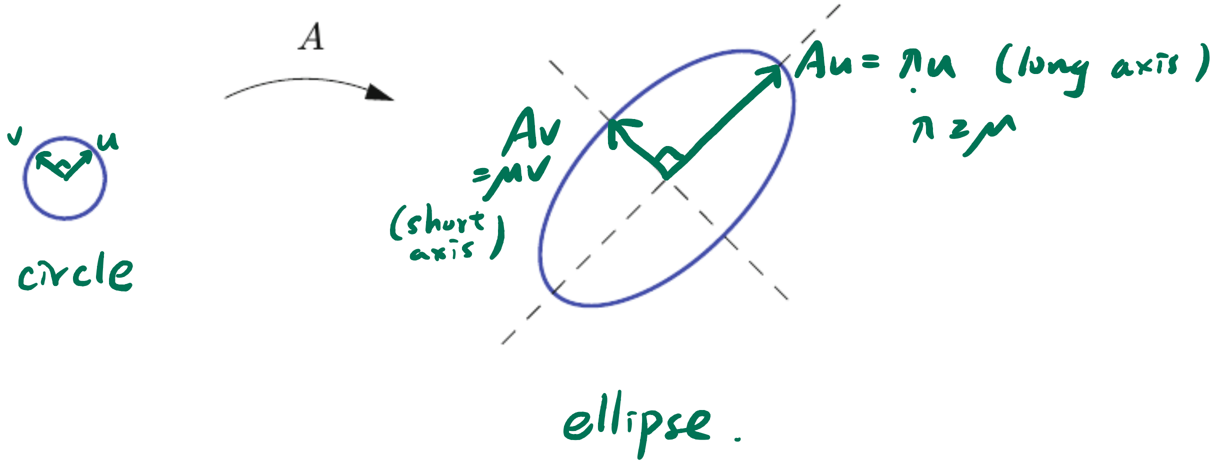
$\langle u, v\rangle = 0$, $\|u\| = 1 = \|v\|$

Suppose that A is a symmetric 2×2 matrix. Let $\mathbf{u}, \mathbf{v}$ be the orthonormal eigenvector basis in $\mathbb{R}^2$ with eigenvalues λ, μ.

$Au = \lambda u$

$Av = \mu v$

Spectral factorization of A: $A = [u\ v]\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}[u\ v]^T$



The matrix A maps the unit circle in $\mathbb{R}^2$ to the ellipse

$$\left\{ a\mathbf{u} + b\mathbf{v} : \frac{a^2}{\lambda^2} + \frac{b^2}{\mu^2} = 1 \right\}$$

The principal directions of this ellipse are the eigenvectors $\mathbf{u}$ and $\mathbf{v}$, and the principal stretches are the eigenvalues λ and μ.

Example 5. Consider the quadratic form

$$q(\mathbf{x}) = 3x_1^2 + 2x_1x_2 + 3x_2^2. = \vec{x}^T A \vec{x} = [x_1, x_2]\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$= a y_1^2 + b y_2^2$

Using the spectral factorization to diagonalized this quadratic form.

$A = A^T$. Find eigenvalues / eigenvectors of A $(A = QDQ^T)$.

$0 = \det(A - \lambda I) \Rightarrow \lambda = 4, 2.$

$\underline{\lambda = 4}$: $\ker(A - 4I)$: $\left(A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right)$ has

a basis $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{\text{normalize}} u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

$\underline{\lambda = 2}$: $\ker(A - 2I)$: $\left(A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$ has

a basis $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \xrightarrow{\text{normalize}} u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

The spectral factorization of A

$$A = Q\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}Q^T, \quad Q = [u_1 \ u_2]$$

$$q(x) = x^T A x = x^T Q \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \underline{Q^T x}$$

$$= y^T \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} y$$

$$= 4y_1^2 + 2y_2^2. \#$$

$\boxed{y = Q^T x}$

coord. of x in basis $\{u_1, u_2\}$.

Then q is diagonalized as $\underline{4y_1^2 + 2y_2^2}$. #

$\underline{\text{RK}}$: $q(\pm u_1) = u_1^T Q \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} Q u_1 = 4.$

$q(\pm u_2) = \quad \cdots \quad = 2.$

§ Optimization principles for eigenvalues of symmetric matrices

Recall: A $n \times n$ symmetric matrix A has real eigenvalues

$$\lambda_1 \geq \cdots \geq \lambda_n$$

and has an orthonormal eigenvector basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Its spectral factorization is

$$A = QDQ^T.$$

Consider the associated quadratic form: for any $\|\mathbf{x}\|_2 = \sqrt{x_1^2 + \ldots + x_n^2} = 1$,

$$q(\mathbf{x}) = \langle A\mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^T A\mathbf{x}.$$

$$= x^T Q D Q^T x \qquad \Big\} \; y = Q^T x$$

$$= y^T D y$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

$$\leq \lambda_1 y_1^2 + \lambda_1 y_2^2 + \cdots + \lambda_1 y_n^2$$

$$= \lambda_1 (y_1^2 + \cdots + y_n^2)$$

$$= \lambda_1$$

Old HW: Q is orthogonal $\|Qx\|_2 = \|x\|_2$

Then $\|y\|_2 = \|Q^T x\|_2 = \|x\|_2 = 1$.

On the other hand, $q(x) = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 \geq \lambda_n (y_1^2 + \cdots + y_n^2) = \lambda_n$.

Thus, we have the result:

Fact 5: Suppose that a symmetric matrix A has real eigenvalues

$$\lambda_1 \geq \cdots \geq \lambda_n.$$

Then

$$\lambda_1 = \max\{\underset{=\, x^T A x}{\langle A\mathbf{x}, \mathbf{x} \rangle} : \|\mathbf{x}\|_2 = 1\}, \qquad \lambda_n = \min\{\underset{=\, x^T A x}{\langle A\mathbf{x}, \mathbf{x} \rangle} : \|\mathbf{x}\|_2 = 1\}.$$

The **maximal value** is achieved when $\mathbf{x} = \pm\mathbf{u}_1$, the unit eigenvector associated with the largest eigenvalue λ_1. $q(\pm u_1) = \lambda_1$

The **minimal value** is achieved when $\mathbf{x} = \pm\mathbf{u}_n$, the unit eigenvector associated with the smallest eigenvalue λ_n. $q(\pm u_n) = \lambda_n$

Example 6. Consider the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

$\lambda_1 = 4$ $\qquad\qquad$ $\lambda_2 = 2.$

Find $\underbrace{\max\{\langle A\mathbf{x}, \mathbf{x}\rangle : \|\mathbf{x}\|_2 = 1\}}_{=\lambda_1 = 4}$ and $\underbrace{\min\{\langle A\mathbf{x}, \mathbf{x}\rangle : \|\mathbf{x}\|_2 = 1\}}_{=\lambda_2 = 2}$.

From EX 5, $q(x) = x^T A x = \langle Ax, x \rangle$.

A has eigenvalues 4, 2.

8.7 Singular Values

While we've seen that eigenvectors and eigenvalues are powerful tools for understanding matrices and operators, they have limitations.

1. Only square matrices can have eigenvectors.
2. Not every matrix has a basis of eigenvectors (only complete/diagonalizable matrices do).
3. Even when a basis of eigenvectors exists, unless the matrix is symmetric, this basis will not be orthogonal.

§ Singular value decomposition (SVD)

We study the factorization of a non-square matrix. The technique is widely used in data analysis.

The key observation is that for *any* real matrix $A = A_{m\times n}$ (not necessarily square), the matrices

$$AA^T, \quad A^TA$$

are both real, symmetric matrices (of sizes m-by-m and n-by-n, respectively).

Let's start by reviewing some facts.

Fact 1: Let $A \in M_{m\times n}$ ($m \times n$ real matrices). Then the following are true.

1. A^TA and AA^T are symmetric.
2. $\ker(A^TA) = \ker(A)$.
3. $\mathsf{rank}(A^TA) = \mathsf{rank}(A)$.

$\mathbb{R}^n$ $\xrightarrow{A}$

$\ker A$

$\operatorname{coimg} A$

SVD decomposition.

Suppose that $A \in M_{m\times n}$ ($m \times n$ real matrices) with $\mathsf{rank}(A) = r$.

Since $A^T A$ is symmetric, the Spectral theorem yields that there exist **orthonormal eigenvectors** $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ forming a basis of $\mathbb{R}^n$ with respect to eigenvalues

$$\lambda_1, \dots, \lambda_r \underbrace{\lambda_{r+1}, \dots \lambda_n}_{\textit{zero eigenvalues}} .$$

Orthonormal

$0 \neq \lambda_1 \longrightarrow v_1$

$\vdots \qquad \vdots$ coimg A

$0 \neq \lambda_r \longrightarrow v_r$

$0 = \lambda_{r+1} \longrightarrow v_{r+1}$

$\vdots \qquad \vdots$ ker A

$0 = \lambda_n \longrightarrow v_n$

Then

$$A^T A\mathbf{v}_j = \lambda_j \mathbf{v}_j, \qquad j = 1, \dots, r$$

and

$$A^T A\mathbf{v}_k = 0, \qquad k = r+1, \dots, n$$

Then we have

Fact 2: (We will show it later.)

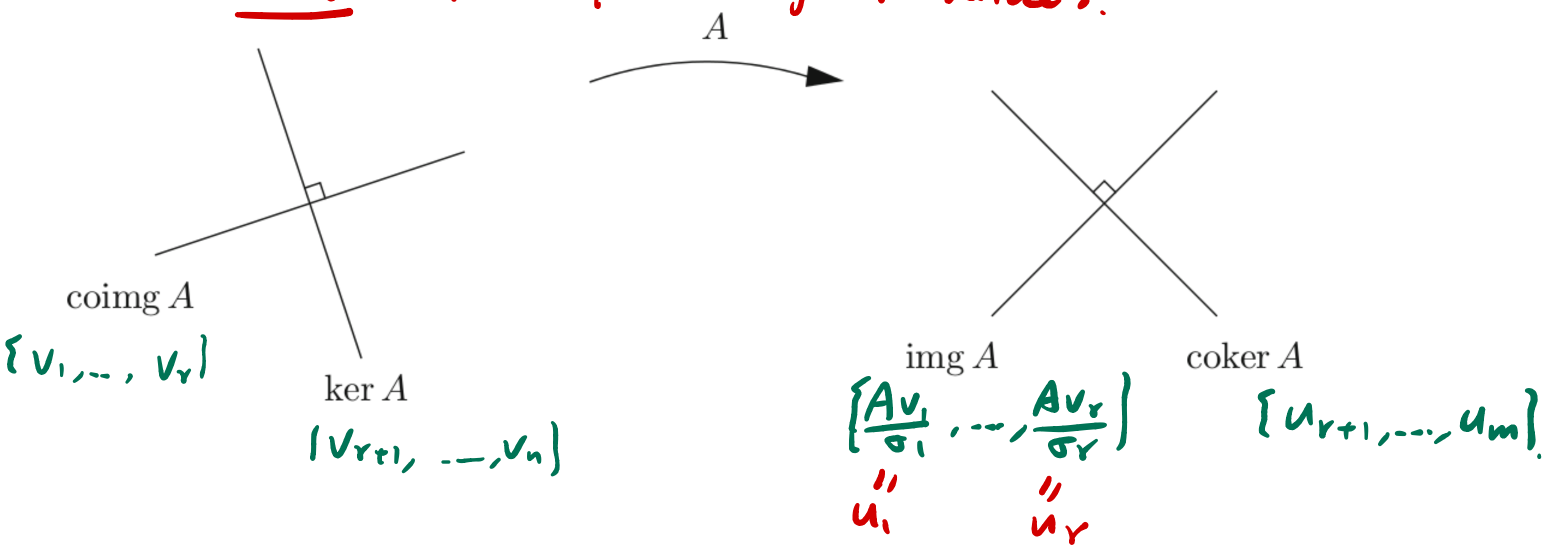
(1) $\lambda_1, \dots, \lambda_r > 0$.

(2) $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is orthonormal basis for coimg A and
$\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$ is orthonormal basis for ker A.

(3) $\{A\mathbf{v}_1, \dots, A\mathbf{v}_r\}$ are mutually orthogonal and, moreover,

$$\left\{\frac{A\mathbf{v}_1}{\sqrt{\lambda_1}}, \dots, \frac{A\mathbf{v}_r}{\sqrt{\lambda_r}}\right\} \quad \text{is orthonormal basis for img } A$$

Denote $\sigma_i = \sqrt{\lambda_i}$, singular values.



Recall: $\dfrac{Av_j}{\sigma_j} = u_j$, $1 \le j \le r$. $\Rightarrow \begin{cases} Av_j = \sigma_j u_j, & 1 \le j \le n \\ Av_j = 0, & r+1 \le j \le n \end{cases}$

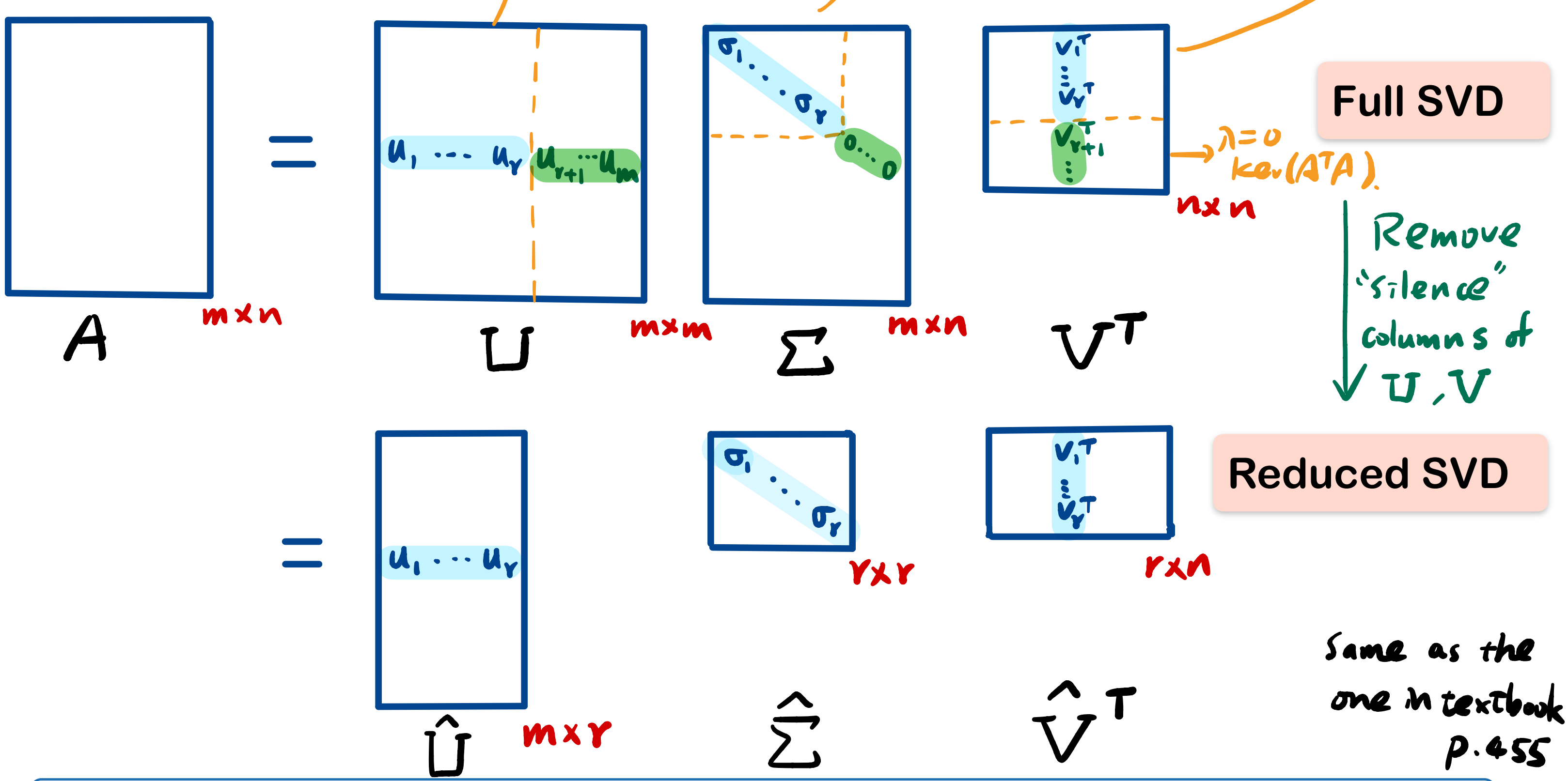
$$A\underbrace{[v_1 \cdots v_r \; v_{r+1} \cdots v_n]}_{V} = [\sigma_1 u_1 \cdots \sigma_r u_r \; 0 \cdots 0]$$

$$= \underbrace{[u_1 \cdots u_r \; u_{r+1} \cdots u_m]}_{U} \overbrace{\begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & 0 \cdots 0 \\ & & 0 & \\ & & \vdots & \\ & & 0 & \end{bmatrix}}^{\Sigma}_{m\times n}$$

- U, V orthogonal matrix since $\{u_i\}$ $\{v_j\}$ are O.N.B.
- v_j : "right" <u>singular vector</u> j
  u_j : "left" " "

$u_j = Av_j/\sigma_j$ $\qquad$ $\sigma_j = \sqrt{\lambda_j}$

$m \ge n$: rank$(A) = r$. $\qquad$ U, V : orthogonal matrices



Same as the one in textbook p.455

Definition: The square roots of the eigenvalues λ_j of A^TA are called the singular values $\sigma_1, \sigma_2, \cdots, \sigma_n$ of an $m \times n$ matrix A. (that is, $\sigma_j = \sqrt{\lambda_j}$)

Poll Question 1: If 5 is an eigenvalue of a square matrix A, then $1/5$ is the eigenvalue of A^{-1}.

A) Yes
B) No