Lecture 38: Quick review from previous lecture

- Let $A \in M_{m\times n}$ ($m \times n$ real matrices) of rank r with positive singular values $\sigma_1 \geq \ldots \geq \sigma_r$. Then
$$\|A\|_F = \sqrt{\sigma_1^2 + \ldots + \sigma_r^2}$$
and
$$\|A\|_2 = \sigma_1 \qquad \text{(largest singular value).}$$
- The **condition number** of a nonsingular $n \times n$ matrix is the ratio between its largest and smallest singular values, namely,
$$\kappa(A) = \frac{\sigma_1}{\sigma_n}.$$

Today we will

- review some concepts

- Lecture will be recorded -

- **Exam 3:** 5/3 (Monday) in lecture.
- Practice Exam is on Canvas now.
- This Friday's office hour is canceled. If you have questions on Friday, we can discuss after the lecture. So HW 13 is extended to Saturday by 6pm.
- **Additional office hours** will be held this Saturday from 10 am-11 am.

Problem 1: Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$. Find A^{-1}.

$$(A \mid I) = \left(\boxed{\begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{matrix}} \;\middle|\; \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right)$$

$$\xrightarrow[③-2①]{②-①} \left(\begin{matrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{matrix} \;\middle|\; \begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{matrix}\right)$$

$$\xrightarrow{② \leftrightarrow ③} \left(\begin{matrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix} \;\middle|\; \begin{matrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ -1 & 1 & 0 \end{matrix}\right)$$

$$\xrightarrow{-②} \left(\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \;\middle|\; \begin{matrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{matrix}\right). \xrightarrow{①-②} \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \;\middle|\; \underbrace{\boxed{\begin{matrix} -1 & 0 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{matrix}}}_{= A^{-1}.}\right)$$

Problem 2: Solve the system $A\mathbf{x} = \mathbf{b}$, where A is the same matrix from Problem 1, and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

$$x = A^{-1}b = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \#$$

$(A \mid b) \longrightarrow$ to solve x

Normal Equation: $A^TAx = A^Tb$.

If A^TA is nonsingular, then the least squares solution $x^* = (A^TA)^{-1}A^Tb$.

Problem 3: Find all solutions to the linear system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$\vec{x} = (x, y, z, \underline{w})^T$ free variable.

$$(A \mid b) \xrightarrow{③-①} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right)$$

$z = 0$

$y = w$

$x = -y - w = -2w.$

General solutions are $\begin{pmatrix} -2w \\ w \\ 0 \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} w, \quad w \in \mathbb{R}.$ #

Problem 4: Suppose A and B are n-by-n matrices and B is orthogonal. If $\det(A) = -2$, what is $\det(AB)$?

$$\det(AB) = \det A \det B$$
$$= -2\ (\pm 1)$$
$$= \underline{\pm 2} \quad \#$$

① $\det(cA) = c^n \det A$, $A_{n\times n}$.

② $\det(AB) = \det A \det B$.

③ Q is orthogonal

$Q^TQ = QQ^T = I$.

ⓐ $Q^T = Q^{-1}$

ⓑ $Q = [v_1 \cdots v_n]_{n\times n}$

$\{v_i\}$ is orthonormal basis of $\mathbb{R}^n$.

ⓒ $\det Q = \pm 1$.

Problem 5:

a) Find the symmetric 3-by-3 matrix K satisfying quadratic form

$$q(\mathbf{x}) = \mathbf{x}^T K \mathbf{x} = x_1^2 + 4x_1x_2 + x_2^2 + 2x_3^2, \quad \text{for all vectors } \mathbf{x} = (x_1, x_2, x_3)^T.$$

$4/2 = 2$ $\sim a y_1^2 + b y_2^2 + c y_3^2$

$$K = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad 3\times 3$$

b) Find the spectral factorization of K. [$K = QDQ^T$, Q is orthogonal

$$0 = \det(K - \lambda I) = \det\begin{pmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

$$= (2-\lambda)\det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (2-\lambda)\left((1-\lambda)^2 - 4\right) \Rightarrow \lambda = 3, 2, -1.$$

$\lambda = 3$: Find $v_1 \in \ker(K - 3I)$: $K - 3I = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

$$\xrightarrow{②+①} \begin{bmatrix} -2 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\text{normalize}} q_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda = 2$: $K - 2I = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $q_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$\lambda = -1$: $K + I = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. $v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \longrightarrow q_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

Spectral Factorization is

$$K = \underbrace{[q_1\ q_2\ q_3]}_{Q} D [q_1\ q_2\ q_3]^T, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \#$$

c) Diagonalize this quadratic form in (a).

$$q(x) = x^T K x = x^T Q D Q^T x$$

$$= y^T D y \qquad y = Q^T x$$

$$= 3y_1^2 + 2y_2^2 - y_3^2. \#$$

d) Find all eigenvalues of K and use that to determine if K is positive definite.

NO.

[$K > 0$ iff all eigenvalues are positive]

Problem 6: Prove that if A is any matrix, then AA^T and A^TA are both symmetric matrices.

$$(AA^T)^T = (A^T)^T A^T = AA^T.$$ So AA^T is symmetric

Similarly,

$$(A^TA)^T = A^T A.$$ #

Problem 7:

a) Compute $\|\mathbf{x}\|_2$, where $\mathbf{x} = (1, 2, 3)^T$. $= \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$.

eigenvalues $-4, -1, 7$;

singular values. $4, 1, 7$.

b) Find the natural matrix norm of $A = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$, with respect to the standard Euclidean norm $\|\mathbf{y}\|_2 = \sqrt{y_1^2 + y_2^2 + y_3^2}$ on $\mathbb{R}^3$.

$$\|A\|_2 = \max\{|-4|, |-1|, |7|\} = 7.$$

RK: $\|A\|_F = \sqrt{(-4)^2 + (-1)^2 + 7^2}$.

Problem 8: Find all vectors in $\mathbb{R}^3$ orthogonal to both $(1,2,0)^T$ and $(0,1,2)^T$ with respect to usual inner product.

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} z, \quad \forall\ z \in \mathbb{R}. \quad \#.$$

Problem 9: Define the operator $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L[\mathbf{v}] = A\mathbf{v}$, where $A = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$. Find the matrix representation for L in the basis $\underset{v_1}{(1,0)^T}, \underset{v_2}{(2,1)^T}$ for both domain and codomain.

$$\{e_1, e_2\} \xrightarrow{A} \{e_1, e_2\}.$$

$$\{v_1, v_2\} \xrightarrow{B} \{v_1, v_2\}$$

$B = S^{-1}AS$, where $S = [v_1 \, v_2]$.

$$B = S^{-1}AS = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 4 & 7 \\ -1 & -1 \end{bmatrix}$$

Problem 10: Write down the 2-by-2 matrix A satisfying $A\mathbf{v}_1 = \mathbf{w}_1$ and $A\mathbf{v}_2 = 2\mathbf{w}_2$, where $\mathbf{v}_1 = (1,1)^T$, $\mathbf{v}_2 = (-1,1)^T$, $\mathbf{w}_1 = (1,1)^T$, and $\mathbf{w}_2 = (-2,-2)^T$.

$$A \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} w_1 & 2w_2 \end{bmatrix}$$

$$A = \begin{bmatrix} w_1 & 2w_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

$$= \frac{1}{2} \begin{bmatrix} 5 & -3 \\ 5 & -3 \end{bmatrix}. \quad \#$$

Problem 11: Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$. Clearly indicate which eigenvector belongs to each eigenvalue. Then diagonalize the matrix.

To be continued!

Problem 12: Find a 2-by-2 matrix A with eigenvalues 2 and -3 and corresponding eigenvectors $(1,-1)^T$ and $(1,0)^T$.

$/\!/ v_1$ $/\!/ v_2$

$$A \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 & -3v_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

$\searrow$ $\underline{VDV^{-1}}$ #

$$= \begin{bmatrix} -3 & -5 \\ 0 & 2 \end{bmatrix}. \quad \#$$

Problem 13: Find a 2-by-3 matrix having rank 1 whose singular value is 2, left singular vector is $(1,2)^T/\sqrt{5}$, and right singular vector is $(1,0,1)^T/\sqrt{2}$.

To be continued!