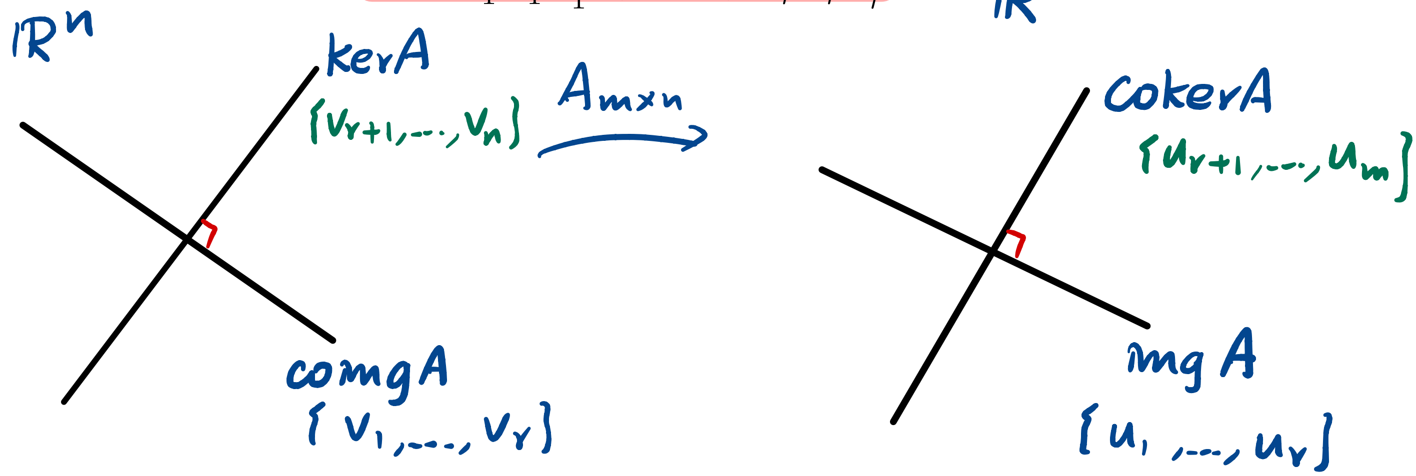
Lecture 39: Quick review from previous lecture

- Let $A \in M_{m\times n}$ ($m\times n$ real matrices) of rank r. Suppose A has positive singular values $\sigma_1 \ge \ldots \ge \sigma_r$ and corresponding right singular vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ and left singular vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$. Then

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \ldots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T.$$



$u_i = \dfrac{A v_i}{\sigma_i}, \quad 1 \le i \le r$

$$A = \begin{bmatrix} u_1 \cdots u_r & u_{r+1} \cdots u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \; 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \\ v_{r+1}^T \\ \vdots \\ v_n^T \end{bmatrix} \quad \text{(Full SVD)}$$

Today we will

- review some concepts

$$= \begin{bmatrix} u_1 \cdots u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}_{r\times r} \begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \end{bmatrix} \quad \text{(Reduced SVD)}$$

- Lecture will be recorded -

$$= \sigma_1 u_1 v_1^T + \ldots + \sigma_r u_r v_r^T \quad . \#$$

- **Exam 3:** 5/3 (Monday) in lecture.
- Practice Exam is on Canvas now.
- This Friday's office hour is moved to the time slot after today's lecture. So HW 13 is extended to Saturday by 6pm.
- **Additional office hours** will be held this Saturday from 10 am-11 am.

Problem 10: Write down the 2-by-2 matrix A satisfying $A\mathbf{v}_1 = \mathbf{w}_1$ and $A\mathbf{v}_2 = 2\mathbf{w}_2$, where $\mathbf{v}_1 = (1,1)^T$, $\mathbf{v}_2 = (-1,1)^T$, $\mathbf{w}_1 = (1,1)^T$, and $\mathbf{w}_2 = (-2,-2)^T$.

[We have discussed in Lecture 38]

Problem 11: Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$. Clearly indicate which eigenvector belongs to each eigenvalue. Then diagonalize the matrix.

$A = VDV^{-1}$

D: diagonal

V: nonsingular.

$0 = \det(A - \lambda I)$. $\lambda = 1, -2$.

$\lambda = 1$: $A - I = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}$. $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\xrightarrow{②+①} \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$.

$\lambda = -2$: $A + 2I = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1}.$$ #

Q: Let $L[\mathbf{v}] = A\mathbf{v}$. Find the matrix representation of L in a basis consisting of eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2\}$.

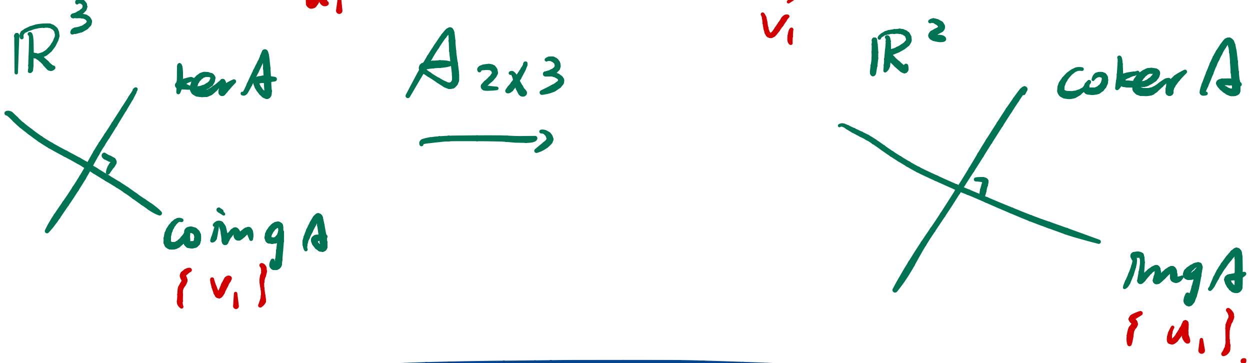
$B = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$. #.

Problem 12: Find a 2-by-2 matrix A with eigenvalues 2 and -3 and corresponding eigenvectors $(1,-1)^T$ and $(1,0)^T$.

[We have discussed in Lecture 38]

Problem 13: Find a 2-by-3 matrix having rank 1 whose singular value is 2, left singular vector is $(1,2)^T/\sqrt{5}$, and right singular vector is $(1,0,1)^T/\sqrt{2}$.

$= u_1$ $\quad$ $= v_1$



$$A_{2\times 3} = [u_1]\,[2]_{1\times 1}\,[v_1^T].$$

$$= \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \underline{[2]\,\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}$$

$$= \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \begin{bmatrix} \frac{2}{\sqrt{2}} & 0 & \frac{2}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{10}} & 0 & \frac{2}{\sqrt{10}} \\ \frac{4}{\sqrt{10}} & 0 & \frac{4}{\sqrt{10}} \end{bmatrix}. \quad \#$$

$$A = 2\,u_1 v_1^T.$$

Problem 14: Write out the full and reduced SVD of the matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$.

① $A^TA = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.

$0 = \det(A^TA - \lambda I)$. $\lambda = 4, 0$.

$\sigma_1 = 2, \quad \sigma_2 = 0.$

② $\underline{\lambda = 4}$: $A^TA - 4I$. $v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (so that $\|v_1\|_2 = 1$)

$\underline{\lambda = 0}$: $v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Find $u_2 \perp u_1$: $u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$A = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}$ (Full SVD)

$= \begin{bmatrix} u_1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} v_1^T \end{bmatrix}$ (Reduced SVD)

$$\boxed{A = 2 u_1 v_1^T}$$

$\sigma_1 = 7, \ \sigma_2 = 3, \ \sigma_3 = 1.$

Problem 15: Suppose A is a 3-by-3 symmetric matrix with eigenvalues $1, 3, -7$. Find the operator norm of A and the Frobenius norm of A.

$$\|A\|_2 = 7$$

$$\|A\|_F = \sqrt{7^2 + 3^2 + 1^2} = \sqrt{59}.$$

① $\|A\|_2 = \sigma_1$

② $\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$ if $\operatorname{rank} A = r$

③ $A = A^T$:

$\sigma_i = |\lambda_i|.$

Problem 16: Suppose A has characteristic polynomial $p_A(\lambda) = \lambda^2 - 2\lambda + 7$. Find the determinant of A and the trace of A. (A has 2 eigenvalues λ_1, λ_2)

$$p_A(\lambda) = \det(A - \lambda I)$$

$$= (\lambda - \lambda_1)(\lambda - \lambda_2)$$

$$= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

$\lambda_1 + \lambda_2 = 2 = \operatorname{tr}(A);$

$\lambda_1 \lambda_2 = 7. = \det A.$

$\operatorname{tr} A = \sum \lambda_i$

$\det A = \lambda_1 \cdots \lambda_n.$

Problem 17: Suppose $A = A^T$ is a symmetric 2-by-2 matrix, and $\det A = 6$. Suppose that $A\mathbf{v} = 2\mathbf{v}$, where $\mathbf{v} = (1,1)^T$. Write the spectral factorization of A.

$A = QDQ^T$

Q: orthogonal!

$6 = \det A = 2 \cdot 3$

$v = \begin{pmatrix}1\\1\end{pmatrix}$ $\quad\langle v_2, v_1\rangle = 0$ (since $A = A^T$).

$v_2 = \begin{pmatrix}1\\-1\end{pmatrix}.$

normalize v, v_2: $q_1 = \frac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix}$, $q_2 = \frac{1}{\sqrt2}\begin{pmatrix}1\\-1\end{pmatrix}.$

$$A = \begin{bmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{bmatrix}^T.$$

Problem 18: Let $\mathcal{P}^{(n)}$ be the space of polynomials of degree $\leq n$.

a) Let $L[p](x) = \int_0^x p(t)dt$ denote the integration operator. Find the matrix representation of L in the monomial bases of $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$.

$\{x, 1\}$ $\quad$ $\{x^2, x, 1\}$.

$$L[x] = \int_0^x t\,dt = \frac{1}{2}x^2 \rightarrow \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}$$

$$L[1] = x \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

$$L\left[\begin{pmatrix} a \\ b \end{pmatrix}\right] = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

b) Let $T[p](x) = p'(x)$. Find the matrix representation of $T : W \rightarrow V$ in the monomial bases of $\mathcal{P}^{(2)}$ and $\mathcal{P}^{(1)}$.

$ax^2 + bx + c$

$T: P^{(2)}_{\{x^2, x, 1\}} \longrightarrow P^{(1)}_{\{x, 1\}}$

$$T[x^2] = 2x \rightarrow \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$T[x] = 1 \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T[1] = 0 \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$T\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \quad \#$$

Problem 19: Suppose A is a 3-by-3 matrix with singular values $\overset{\sigma_3}{1}, \overset{\sigma_2}{2}$, and $\overset{\sigma_1}{3}$. What is the condition number of A? What are the singular values of A^{-1}? What are the singular values of A^T? What is the determinant of A?

1) $K(A) = \frac{\sigma_1}{\sigma_3} = 3/1$

2) A^{-1}'s singular values : $\frac{1}{3}, \frac{1}{2}, \frac{1}{1}$.

3). A^T " : $3, 2, 1$.

4) $\det A = \det(U \Sigma V^T)$

$= \det U \det \Sigma \det V^T$

$= \pm 1 \det \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \pm 1 = \underline{\pm 6}$.

Problem 20: Suppose A is a 3-by-3 <u>symmetric</u> matrix of the form

$$A = 2\mathbf{u}_1\mathbf{u}_1^T - 3\mathbf{u}_2\mathbf{u}_2^T + 6\mathbf{u}_3\mathbf{u}_3^T, = QDQ^T = Q\begin{bmatrix} 2 & & \\ & -3 & \\ & & 6 \end{bmatrix}Q^T$$

where $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{R}^n$ are nonzero column vector and are <u>orthonormal</u>. What is the condition number of A? What are the singular values of A^{-1}? What are the singular values of A^T? What is the determinant of A?

$A = A^T$. $\sigma_i = |\lambda_i|$. $\sigma_1 = 6$, $\sigma_2 = 3$, $\sigma_3 = 2$.

1) $K(A) = 3$

2) A^{-1} : $1/6, 1/3, 1/2$

3). A^T : $6, 3, 2$.

$\det Q^T = \det Q^{-1} = \frac{1}{\det Q}$

4) $\det A = \det Q \det D \det Q^T = \det D$

$= \det(QDQ^T)$

$= 2 \cdot (-3) \cdot 6$

$= \underline{-36}$

Problem 21: Suppose A is a matrix with singular values 2, 3 and 8. Suppose $\mathbf{u}$ and $\mathbf{v}$ are the left and right singular vectors of A with singular value 8, and let $B = 8\mathbf{u}\mathbf{v}^T$. Find $\|A - B\|_2$ and $\|A - B\|_F$.

$$A = \underline{\underline{8uv^T}} + 3u_2v_2^T + 2u_3v_3^T.$$

$$\underline{B = 8uv^T} \quad (\text{rank 1 appro. of } A).$$

$$A - B = 3u_2v_2^T + 2u_3v_3^T$$ has singular value 3, 2.

$$\|A-B\|_2 = 3$$

$$\|A-B\|_F = \sqrt{3^2+2^2} = \sqrt{13}.$$

Q: Find rank 2 appro. of A: $\underline{8uv^T + 3u_2v_2^T}$

Problem 22: Suppose $A = 2\mathbf{u}\mathbf{v}^T$, where $\mathbf{u} = (1, -1)^T/\sqrt{2}$ and $\mathbf{v} = (1, 1)^T/\sqrt{2}$. Let $\mathbf{b} = (1, 0)^T$. (1) Find the unique vector $\mathbf{x}$ that minimizes $\|A\mathbf{x} - \mathbf{b}\|_2$ and has the smallest Euclidean norm. (2) Find all least squares solutions to $A\mathbf{x} = \mathbf{b}$. That is, find all vectors $\mathbf{x}$ that minimize $\|A\mathbf{x} - \mathbf{b}\|_2$.

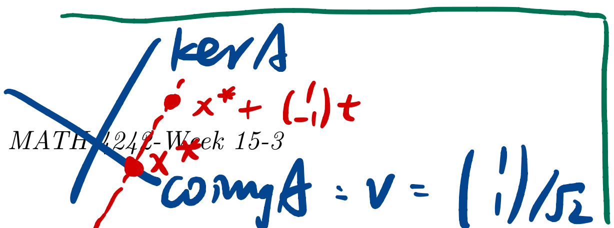
① $A = 2uv^T = [u][2][v^T]$ (Reduced SVD)

Pseudoinverse

$$A^+ = [v][\tfrac{1}{2}][u^T]$$

$$x^* = A^+b = \begin{pmatrix} \frac14 & -\frac14 \\ \frac14 & -\frac14 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac14 \\ \frac14 \end{pmatrix}.$$

② $\operatorname{Ker} A = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} t \;\middle|\; t \in \mathbb{R} \right\}.$ (orthogonal to v)



So all least squares solutions are $x^* + \begin{pmatrix} 1 \\ -1 \end{pmatrix} t = \begin{pmatrix} \frac14 \\ \frac14 \end{pmatrix} + \begin{pmatrix} t \\ -t \end{pmatrix}$, $t \in \mathbb{R}$.