Lecture 4: Quick review from previous lecture

- We have discussed two type of elementary row operators:
  - **type 1** - adding/subtracting one row to /from another row ;
  - **type 2** - permutation (pivoting)
- We learned how to use "LU factorization" to solve a linear system $A\mathbf{x} = \mathbf{b}$.
- If A is **nonsingular**, A has a permuted LU factorization: $PA = LU$

$$L\underbrace{Ux}_{y} = b$$

① $Ly = b$

② $Ux = y$

$$A \xrightarrow{\text{type 1 or 2}} \qquad U = \begin{bmatrix} \diagdown & \\ 0 & \end{bmatrix}$$

Today we will

- continue discuss Sec. 1.4 Pivoting and Permutations
- discuss Sec. 1.5 Matrix Inverse

- Lecture will be recorded -

- The first homework is due this Friday (1/29) at 6pm.

Example 4. Find LU factorization of the matrix

$$A = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 3 & -3 \\ -2 & -6 & -2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}, \quad L = I_4, \quad P = I_4.$$

$$A \xrightarrow[\substack{③+2①\to E_2 \\ ④-①\sim E_3}]{②-2①\to E_1} \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad \underline{P = I_4}$$

$$P_1 \quad \xrightarrow{②\longleftrightarrow④} \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & -7 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

$$P_2 \quad \xrightarrow{③\longleftrightarrow④} \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 5 \end{pmatrix} = U, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Then $PA = LU$

$$P_2 P_1 E_3 E_2 E_1 A = U$$

$$\underbrace{(E_1^{-1} E_2^{-1} E_3^{-1} P_1 P_2)\, P_2 P_1 E_3 E_2 E_1 A}_{A} = (E_1^{-1} E_2^{-1} E_3^{-1} P_1 P_2)\, U$$

$$PA = \underbrace{P}\, \overbrace{(E_1^{-1} E_2^{-1} E_3^{-1} P_1 P_2)}^{L}\, U$$

Remark: Performing 2^{nd} type of elementary row operation (permuting rows) indeed can give: *PA is regular (has all nonzero pivot)*. Then we can find its LU factorization, namely,

$$\boxed{PA = LU}$$

§ **The permuted LU factorization can be used to solve linear systems $A\mathbf{x} = \mathbf{b}$.** $(PA = LU)$. $PAx = Pb. \Rightarrow \boxed{\underbrace{Ux}_{= y} L = Pb}$

1st system : $Ly = \boxed{Pb}$ (solve "y" by using forward substitution)

2nd system : $Ux = y$ (solve "x" by using back substitution)

Example 5. Let A be the matrix from the previous example. Solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

1. $Ly = \underline{\underline{Pb}}$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix}.$$

Using forward substitution,

① $y_1 = 1$

② $y_1 + y_2 = 4 \Rightarrow y_2 = 3.$

③ $2\overset{1}{y_1} + y_3 = 2 \Rightarrow y_3 = 0.$

④ $-2(1) + y_4 = 3 \Rightarrow y_4 = 5$

2. $Ux = y = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix}.$

[Continue] $\begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 5 \end{bmatrix} x = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix}$.

Using back substitution,

(4) $x_4 = 1$

(3) $x_3 - 7x_4 = 0$, $x_3 = 7$.

(2) $-x_2 + x_4 = 3$, $x_2 = -2$

(1) $x_1 + 3(-2) + 7 + 2(1) = 1$.

$x_1 = -2$.

$x = \begin{pmatrix} -2 \\ -2 \\ 7 \\ 1 \end{pmatrix}$. [Check this solution by pluging $\begin{pmatrix} -2 \\ -2 \\ 7 \\ 1 \end{pmatrix}$ into $Ax = b$.]

Fact 2: If A is square and nonsingular, then $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for any right hand side $\mathbf{b}$.

1.5 Matrix Inverse

The inverse of a matrix is analogous to $a^{-1} = \frac{1}{a}$ of a scalar $a \neq 0$.
Ex:

$$\underbrace{[5]}_{1\times 1\, matrix} \quad \text{has inverse } [\,\tfrac{1}{5}\,].$$

Then

$$[\tfrac{1}{5}][5] = [5][\,\frac{1}{5}\,] = [1].$$

Definition: A is an $n \times n$ matrix. Then we call an $n \times n$ matrix X the **inverse** of A if it satisfies

$$A\overbrace{X}^{\text{right inverse}} = I_n = \overbrace{X}^{\text{left inverse}}A.$$

We then denote such matrix X (**inverse** of A) by A^{-1}.

Note that "Not every matrix has an inverse !!!"

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. There is NO (2×2) matrix X satisfying

$$XA = I_2 = AX.$$

In fact, we will see a square matrix has an inverse when it is nonsingular.

We've already seen how to find the inverse of elementary row matrices:

Example 1. Find the inverse of the following matrix :

(1) $E = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. , elementary matrix "Add 2 ② to ①".

$$E^{-1} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$E^{-1}E = EE^{-1} = I_3.$$

(2) A permutation matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. elementary matrix "permute ① ②".

$$PP = I_3$$

Then $\boxed{P^{-1} = P}$.

In general, however, finding A^{-1} will NOT be so easy.

We will see a systematic method for doing so in the next class, known as **Gauss-Jordan elimination**.

In the following, we will discuss 3 key facts:

Fact 1. If the inverse of a matrix exists, then this inverse matrix is unique. In other words, if B and C are both inverse of A, then

$$B = C.$$

Proof: B, C are inverse of A. By definite

$$BA = I = AB$$
$$CA = I = AC.$$

$$B = BI = B(AC) \overset{\text{associate property}}{=} (BA)C = IC = \underline{\underline{C}}.$$

Fact 2. The inverse of the inverse is the original matrix. More precisely, $(A^{-1})^{-1} = A$.

Proof. This is an immediate consequence of the defining property of A^{-1}.

$$\Big[\ \underbrace{A}_{= (A^{-1})^{-1}} A^{-1} = I = A^{-1} A.\ \Big]$$

means it has inverse

Fact 3. If A and B are two invertible n-by-n matrices, then their product AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$. (sometimes called "shoes and socks theorem".

Proof: Let $X = B^{-1}A^{-1}$.

To see if X is the inverse of AB, then we only need to check

$X(AB) = I$, $(AB)X = I$.

1. $X(AB) = I$:

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}\,I\,B = B^{-1}B = I. \ \#$$

2. $(AB)X = I$:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A\,I\,A^{-1} = AA^{-1} = I. \ \#$$

Remark: In general,

$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}.$$

§ The inverse of a 2×2 matrix.

Consider 2-by-2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

If $ad - bc \neq 0$, then A has an inverse given by:

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The number "$ad - bc$" is known as the determinant of A, denoted by

$$\boxed{\det(A) = ad - bc}$$

In general, the determinant $\det(A)$ can be defined for a square matrix A of any size [*Will discussed in later lectures*], and

$$A \text{ is invertible if and only if } \det(A) \neq 0.$$

Example 2: (1) Find the inverse of $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$.

To be continued !

(2) Is the matrix $A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$ invertible?

Poll Question 1: If A, B, C are invertible $n \times n$ matrices, then

$A)$ $(ABC)^{-1} = C^{-1} + B^{-1} + A^{-1}$
$B)$ $(ABC)^{-1} = \underline{C^{-1}B^{-1}A^{-1}}$
$C)$ $(ABC)^{-1} = A^{-1}B^{-1}C^{-1}$

* You should be able to see the pop up Zoom question. Answer the question by clicking the corresponding answer and then submit.

Caution: after clicking submit, you will not be able to resubmit your answer!