Lecture 5: Quick review from previous lecture

- We talked about what is the inverse of a given square matrix $\overset{n\times n}{A}$, that is, if A is a square matrix, then we call $n \times n$ matrix X the **inverse** of A if

$$\underbrace{AX}_{\text{right inverse}} = I_n = \underbrace{XA}_{\text{left inverse}}.$$

  We denote X(inverse of A) by A^{-1}.

- If the inverse of a matrix A exists, then this inverse matrix is "unique". We call A is "**invertible**".

Today we will

- continue discuss Sec. 1.5 Matrix Inverse

- Lecture will be recorded -

- The first homework is due **Today** (1/29) at 6pm.

§ The inverse of a 2×2 matrix.

Consider 2-by-2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

If $ad - bc \neq 0$, then A has an inverse given by:

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The number "$ad - bc$" is known as the determinant of A, denoted by

$$\boxed{\det(A) = ad - bc}$$

In general, the determinant $\det(A)$ can be defined for a square matrix A of any size [*Will discussed in later lectures*], and

$$A \text{ is invertible if and only if } \det(A) \neq 0.$$

Example 2: (1) Find the inverse of $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$.

$\det A = 2 \cdot 5 - 3 \cdot 4 = 10 - 12 = -2 \neq 0.$

$A^{-1} = \frac{1}{-2} \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}.$

Check: $A^{-1} A = I_2 = A A^{-1}.$

(2) Is the matrix $A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$ invertible? (No)

$\det A = 2 \cdot 6 - 12 = 0.$

Observation: 2nd row of A = 2 (1st row).

row ② & row ① are parallel.

§ **The inverse of a 2×2 matrix.**

If you are interested in how we get the formula of the inverse of a 2×2 matrix above, the explanation/computations is as follows: (See also Example 1.15 in page 31 in Textbook)

Consider a general 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let's compute its inverse $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ if it exists. (that is, check $XA = I$ $AX = I$).

(1) $AX = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{cases} ax + bz = 1 \\ ay + bw = 0 \\ cx + dz = 0 \\ cy + dw = 1 \end{cases} \Longrightarrow \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Using Gaussian-Elimination to solve 4×4 linear system:

__Case 1:__ $a \neq 0$

$$\xrightarrow{③ - \frac{c}{a}①} \left(\begin{array}{cccc|c} a & 0 & b & 0 & 1 \\ 0 & a & 0 & b & 0 \\ 0 & 0 & d - bc/a & 0 & -c/a \\ 0 & c & 0 & d & 1 \end{array}\right)$$

$$\xrightarrow{④ - \frac{c}{a}②} \left(\begin{array}{cccc|c} a & 0 & b & 0 & 1 \\ 0 & a & 0 & b & 0 \\ 0 & 0 & d - bc/a & 0 & -c/a \\ 0 & 0 & 0 & d - bc/a & 1 \end{array}\right)$$

By Back substitution, we get

$$\begin{cases} w = \dfrac{a}{ad-bc} \quad \text{if } ad-bc \neq 0 \\ A \text{ has } \underline{\text{NO}} \text{ inverse if } ad-bc = 0. \end{cases}$$

[Continue the computation:]

Thus $\boxed{x = \frac{d}{ad-bc}, \ y = \frac{-b}{ad-bc}, \ z = \frac{-c}{ad-bc}, \ w = \frac{a}{ad-bc}}$

if $ad-bc \neq 0$.

Case 2 : $a = 0$.

Suppose also $c \neq 0$.

Similar computations give the same formula

if $ad-bc \neq 0$.

Case 3 : $a=0$, $c=0$.

Then A has NO inverse. #

1.5 Matrix Inverse (Continue ...)

§ Introduction to Gauss-Jordan Elimination.

We have discussed two type of elementary row operators:

- type 1 - adding/subtracting one row to another row ;
- type 2 - permuting the order of rows (pivoting).

Now for **Gauss-Jordan Elimination**, in addition to type 1 and type 2 row operators above, we will use the 3^{rd} elementary row operator, that is,

- type 3 - scaling a row of A by a nonzero multiple.

Note that "In the linear systems, multiplying one equation by a non-zero number obviously does NOT change the solution to the system."

Like the other elementary row operations, we have

Example 3: The **elementary matrix** that associated to the "scales the 2^{nd} row by 8 is":

$$I_2 \xrightarrow{8(2)} \qquad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let A be the 3-by-4 matrix

$$A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{pmatrix}$$

Then

$$EA = \begin{pmatrix} a & b & c & d \\ 8e & 8f & 8g & 8h \\ i & j & k & l \end{pmatrix}$$

§ Gauss-Jordan elimination

The goal is to find the matrix X (the inverse of A) satisfying

$$AX = XA = I$$

Before we start, let's recall that

$$[B]_{n\times m}\,[A]_{m\times n} = I_n$$

Let A be a $m \times n$ matrix. Generally,
we say $n \times m$ matrix B is a **left** inverse of A if $\boxed{BA = I_n}$; and
we say $n \times m$ matrix C is a **right** inverse of A if $\boxed{AC = I_m}$.

$$[A]_{m\times n}\,[C]_{n\times m} = I_m.$$

We also state a fact below:

Fact 4: Let A be a $n \times n$ **(square)** matrix. If X is the **right** inverse of A, then such X is automatically be the **left** inverse of A.

The **Gauss-Jordan Elimination** is to perform elementary row operations:

- type 1 - adding/subtracting one row to another row ;
- type 2 - permuting the order of rows (pivoting);
- type 3 - scaling a row of A by a nonzero multiple.

to A and then

$$\boxed{\text{turn } A \text{ into } I \text{ (the identity matrix),}}$$

if that is possible.

$$A \xrightarrow{E_i} I$$

Then we would have

$$(E_m E_{m-1} \cdots E_2 E_1)A = I,$$

where $E_1, \ldots, E_m$ is elementary matrix associated to type 1 or type 2 or type 3.

Then

$$A^{-1} = E_m E_{m-1} \cdots E_2 E_1.$$

§ The operations to convert A to I are broken into 3 stages.

(1) bring $A \to$ upper triangular form;

(2) divide each row of A by the corresponding pivot (i.e. that row's diagonal element)

(3) More row operations to clear out the elements above the diagonal of A, and turn it into the identity.

$$A \xrightarrow{(1)} \begin{bmatrix} \ddots & \\ 0 & \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & \\ 0 & 1 \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example 4. Find the inverse A^{-1} of

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 3 \end{pmatrix}$$

argumented matrix.

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 0 & -1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right).$$

$$\xrightarrow{① \leftrightarrow ③} \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \end{array}\right).$$

$$\xrightarrow{② - 2①} \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 0 & 0 & 1 \\ 0 & -1 & -6 & 0 & 1 & -2 \\ 0 & -1 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{③ - ②} \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 0 & 0 & 1 \\ 0 & -1 & -6 & 0 & 1 & -2 \\ 0 & 0 & 6 & 1 & -1 & 2 \end{array}\right)$$

[Example continue...]

$$\xrightarrow{-1 ② ,\ \frac{1}{6} ③} \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 0 & 0 & 1 \\ 0 & 1 & 6 & 0 & -1 & 2 \\ 0 & 0 & \boxed{1} & 1/6 & -1/6 & 1/3 \end{array}\right)$$

$$\begin{array}{c} ② - 6③ \\ \xrightarrow{\qquad\qquad} \\ ① - 3③ \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1/6 & -1/6 & 1/3 \end{array}\right)$$

$$\xrightarrow{① - ②} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\boxed{\begin{matrix} 1/2 & 1/2 & 0 \\ -1 & 0 & 0 \\ 1/6 & -1/6 & 1/3 \end{matrix}}}_{= A^{-1}} \\ 0 & 0 & 1 & \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ -1 & 0 & 0 \\ 1/6 & -1/6 & 1/3 \end{pmatrix}. \quad \#$$

Check $A^{-1}A = I_3$.

$$\left[\ Ax = A(A^{-1}b) = Ib = b\ \right]$$

Fact 5: If matrix A is **nonsingular**, then $\boxed{x = A^{-1}b}$ is the unique solution to the linear system $Ax = b$.

Example 5. Let $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Solve the system $A\mathbf{x} = \mathbf{b}$. with same A above.

$$x = A^{-1}b = \begin{pmatrix} 1/2 & 1/2 & 0 \\ -1 & 0 & 0 \\ 1/6 & -1/6 & 1/3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \\ -1 \\ 1/2 \end{pmatrix}. \quad \#$$

§ Turn to diagonal matrices.

Let $D = \text{diag}(d_1, \ldots, d_m)$ is an m-by-m diagonal matrix. $\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_m \end{bmatrix}$

1. DA is equal to A with the i^{th} row scaled by d_i.

Example 6.

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad A = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$$

d_1, d_2, d_3.

Then $DA = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} = \begin{pmatrix} -a & -d \\ 2b & 2e \\ 4c & 4f \end{pmatrix}.$

2. D is invertible if all of its diagonal entries are non-zero. $(d_i \neq 0)$

Example 7. Same matrix D as above, find D^{-1}.

$$D^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}.$$

<u>Check</u> $D^{-1}D = I_3$

3 . Let D_1 and D_2 be 2 diagonal matrices. Then so is $D_1 D_2$.

$$\underset{n\times n}{\begin{pmatrix} a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_n \end{pmatrix}} \underset{n\times n.}{\begin{pmatrix} b_1 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & b_n \end{pmatrix}} = \underset{n\times n.}{\begin{pmatrix} a_1b_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1}b_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_nb_n \end{pmatrix}}$$

§ LDV factorization

Definition: When a triangular matrix has all 1's on its diagonal, we say it is **unitriangular**.

We already know if A is a **regular** matrix, we can write

$$\boxed{A = LU}, \quad = \underbrace{\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} & & \\ & & \\ 0 & & \end{bmatrix}}_{U}$$

- U : upper triangular with non-zero diagonal elements (the pivots)
- L : lower **uni**triangular, meaning it is lower triangular with all diagonal elements equal to 1

Then turn U into

$$U = \underline{DV} = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \begin{bmatrix} 1 & & \\ & \ddots & \\ 0 & & 1 \end{bmatrix}.$$

where D is diagonal matrix and matrix V is upper unitriangular.

Example 8.

Let $A = LU$, where $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 5 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 3 & 1 & 0 \\ 0 & -2 & 8 \\ 0 & 0 & 7 \end{pmatrix}$.

Find LDV factorization of A.

$$U = \underbrace{\begin{bmatrix} 3 & & 0 \\ & -2 & \\ 0 & & 7 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{V}$$

To be continued!

Fact 6: If A is **nonsingular**, we can form the permuted LDV factorization

$$PA = LDV,$$

where P is a permutation matrix, L is lower **uni**triangular, D is diagonal, and V is upper **uni**triangular.

Poll Question 1: Let $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Then $D^{-1} =$

$A) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$; $\quad B) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{pmatrix}$.

Poll Question 2: Consider the permutation matrix $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Then $P^{-1} =$

$A) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$; $\quad B) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

* You should be able to see the pop up Zoom question. Answer the question by clicking the corresponding answer and then submit.

Caution: after clicking submit, you will not be able to resubmit your answer!