Lecture 6: Quick review from previous lecture

- We talked about what is the inverse of a given square matrix A, that is, if A is a square matrix, then its **inverse** A^{-1} is the $n \times n$ matrix satisfying

$$AA^{-1} = I_n = A^{-1}A.$$

- If A and B are two invertible n-by-n matrices, then their product AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

- If $ad - bc \neq 0$, then the 2-by-2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an inverse given by:

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- If matrix A is **nonsingular**, then $\boxed{\mathbf{x} = A^{-1}\mathbf{b}}$ is the unique solution to the linear system $A\mathbf{x} = \mathbf{b}$.

- **Gauss-Jordan Elimination** to find the inverse of a general square matrix.

$$(A \mid I_n) \xrightarrow[\text{type 2} \atop \text{type 3.}]{\text{type 1}} (I_n \mid A^{-1})$$

Today we will discuss

- 1.6 Transposes and Symmetric Matrices
- 1.8 General linear system

- Lecture will be recorded -

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§ LDV factorization

Definition: When a triangular matrix has all 1's on its diagonal, we say it is **unitriangular**.

We already know if A is a **regular** matrix, we can write

$$\boxed{A = LU},$$

- U : upper triangular with non-zero diagonal elements (the pivots)
- L : lower **uni**triangular, meaning it is lower triangular with all diagonal elements equal to 1

Then turn U into

$$U = DV$$

where D is diagonal matrix and matrix V is upper unitriangular.

Example 8.

Let $A = LU$, where $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 5 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 3 & 1 & 0 \\ 0 & -2 & 8 \\ 0 & 0 & 7 \end{pmatrix}$.

Find LDV factorization of A.

$$= \underbrace{\begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 7 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 1/3 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}}_{V}$$

Fact 6: If A is **nonsingular**, we can form the permuted LDV factorization

$$PA = LU \implies PA = LDV,$$

where P is a permutation matrix, L is lower **uni**triangular, D is diagonal, and V is upper **uni**triangular.

Example 9. Find LDV factorization of

$$A = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 3 & -3 \\ -2 & -6 & -2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

Answer: From Example 4 in Lecture 3, we have known $PA = L\boxed{U}$, where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}, \; U = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

$$U = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{V}$$

$A = [\triangle\ 0]$. $\quad A^{-1} = [\triangle\ 0]$

- A few additional comments; for details, refer to §1.5 in the textbook
- A **triangular matrix** is **nonsingular** if and only if all of its diagonal elements are **non-zero**; see page 39 in the book.
- Any **lower triangular** matrix with all **non-zero** diagonal elements has a **lower triangular inverse**, and any lower unitriangular matrix has a lower unitriangular inverse. Ditto if "lower" is replaced with "upper". Again, see page 39. [1]

[1]**Lower triangular:** all entries above the diagonal are zero.
Lower unitriangular: all entries above the diagonal are zero and entries on diagonal are all 1.

1.6 Transposes and Symmetric Matrices

If A is a matrix of any dimensions, then its **transpose**, denoted A^T, is the matrix that switches the roles of the rows and columns of A.

Example 1. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}_{\underline{4\times 2}}$ then $A^T = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}_{2\times 4}$.

In particular, if A is an m-by-n matrix, then A^T is an n-by-m matrix.

§ Some basic properties of the transposition operations.

1. Taking the transpose twice returns you to the original matrix:

$$\boxed{(A^T)^T = A}$$

<u>EX</u>: $(A^T)^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}_{2\times 4} = A$.

Transposition is compatible with matrix addition and scalar multiplication.

2. If A and B are matrices of the same dimensions and c is a scalar, then:

$$\boxed{(A+B)^T = A^T + B^T}$$

and

$$\boxed{(cA)^T = cA^T}$$

3. When you transpose a product of matrices, it is the product of the transposes in the opposite order:

$$\boxed{(\overset{p\times m}{AB})^T = \overset{p\times m}{B^T A^T}}$$

[Check it:] $A_{m\times n}$, $B_{n\times p}$: AB is $m\times p$ matrix.
$(AB)^T$ is $p\times m$ matrix.

$B^T_{p\times n}$ $A^T_{n\times m}$: $B^T A^T$ is $\underline{p\times m}$ matrix.

4. Transposition commutes with inversion: $\boxed{(A^T)^{-1} = (A^{-1})^T}$.

[Check it:] Let $X = (A^{-1})^T$. We only need to check
$\underline{XA^T = I}$, $A^T X = I$.

$$XA^T = (A^{-1})^T A^T \underset{\text{by 3.}}{=} (AA^{-1})^T = I^T = I$$

#

5. A matrix is said to be **symmetric** if it is equal to its own transpose.

$$\boxed{A = A^T}$$

Example 2. The matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ is symmetric. $(A = A^T)$

*By necessity, symmetric matrices must be square.

$\Longleftrightarrow$

Fact 1. A **symmetric** matrix A is **regular** if and only if

$$\boxed{A = LDL^T} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ \ast & & 1 \end{bmatrix} \begin{bmatrix} \ast & & 0 \\ & \ddots & \\ 0 & & \ast \end{bmatrix} \begin{bmatrix} 1 & & \ast \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

where L is a lower unitriangular matrix and D is a diagonal matrix with nonzero diagonal entries.

* If A is **regular** and $A = LU$, then LU factorization is **unique**. Moreover, its LDV factorization is also **unique**.

[Proof.]

$(\Rightarrow)$ A is symmetric, regular. Show $A = LDL^T$.

Since A is regular, $A = LDV$

$$A^T = (LDV)^T \overset{\text{by 3}}{=} V^T D^T L^T = \underbrace{V^T}_{\text{lower unitriangular}} D \underbrace{L^T}_{\text{upper unitriangular}}$$

$$\| $$

$$A = LD\boxed{V}$$

Since LDV-factorization,

$$V = L^T$$

$$L = V^T$$

Then $A = LDV = LDL^T$. #

$(\Leftarrow)$ $A = LDL^T$. Check $A = A^T$.

$$A^T = (LDL^T)^T = (L^T)^T D^T L^T$$

$$= L D L^T = A. \#$$

- For any nonsingular matrix A, we can use the decomposition $PA = LDV$ to solve a linear system $A\mathbf{x} = \mathbf{b}$. First solve $\boxed{L\mathbf{y} = P\mathbf{b}}$ then $D\mathbf{z} = \mathbf{y}$, then $V\mathbf{x} = \mathbf{z}$.
- Using the LDV or LDL^T decompositions is typically not any easier than using LU for solving linear systems. But writing the symmetric version LDL^T can have some advantages we will see later in the semester.

$$PAx = Pb \implies L\underbrace{D\overbrace{Vx}^{z}}_{y} = Pb$$

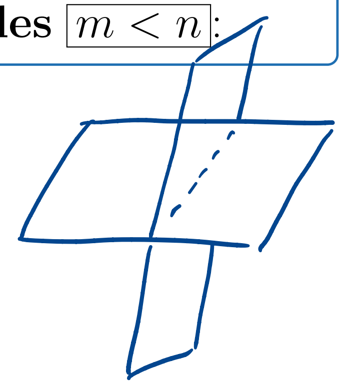
$$\begin{cases} Ly = Pb \\ Dz = y. \\ Vx = z \end{cases}$$

1.8 General linear system

Consider a $m \times n$ matrix A. Here A may be a rectangular matrix or square matrix.

Let's look at the following different situations:

1. **the number of equations < the number of variables** $\boxed{m < n}$:



Example 1. Solve the linear system:

$$\begin{pmatrix} 1 & 2 & 3 \\ -2 & -2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ -2 & -2 & -4 & 4 \end{array}\right) \xrightarrow{②+2①} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 12 \end{array}\right)$$

② $2y + 2z = 12$

$$y = \frac{12-2z}{2} = 6-z$$

① $x + 2(6-z) + 3z = 4$

$$x = 4 - 3z - (12-2z)$$

$$= -8 - z.$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -8-z \\ 6-z \\ z \end{pmatrix}.$$ for any scalar z. #.

infinitely many solutions.