Lecture 7: Quick review from previous lecture

• The transpose of a matrix $A^T$.

• If $A = A^T$, then we call the matrix $A$ is symmetric

• $A$ is nonsingular $\Rightarrow PA = LDV$

• A symmetric matrix $A$ is regular $\iff A = LDL^T$

Here $P$ is a permutation matrix, $L$ is lower unitriangular, $D$ is diagonal, and $V$ is upper unitriangular.

Today we will discuss

• Sec. 1.8 general system.

- Lecture will be recorded -
1.8 General linear system

Consider a $m \times n$ matrix $A$. Here $A$ may be a rectangular matrix or square matrix.

Let’s look at the following different situations:

1. the number of equations $< \text{the number of variables}$ $m < n$:

Example 1. Solve the linear system:

$$
\begin{pmatrix}
1 & 2 & 3 \\
-2 & -2 & -4
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 
\begin{pmatrix}
4 \\
4
\end{pmatrix}
$$

\[ \begin{align*}
(1 & 2 & 3 & | 4) \\
(-2 & -2 & -4 & | 4) \\
\end{align*} \]

\[ \begin{align*}
\begin{pmatrix} 1 & 2 & 3 & | 4 \\ 0 & 2 & 2 & | 12 \end{pmatrix} & \quad \text{①} \\
2y + 2z = 12 & \quad \text{②} \\
y & = \frac{12 - 2z}{2} = 6 - z \\
x + 2(6 - z) + 3z = 4 & \quad \text{①} \\
x & = 4 - 3z - (12 - 2z) \\
& = -8 - z.
\end{align*} \]

\[ \begin{pmatrix} x \\
y \\
z \end{pmatrix} = \begin{pmatrix} -8 - z \\
6 - z \\
\frac{z}{2} \end{pmatrix}. \text{ for any scalar } z.
\]

$\text{\textit{infinitely many solutions}}$
2. the number of equations ≥ the number of variables \( m \geq n \):

Example 2. Solve the linear system:

\[
\begin{bmatrix}
1 & 2 \\
3 & 2 \\
0 & 4
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
4 \\
2 \\
-2
\end{bmatrix}
\]

```
A

\begin{pmatrix}
1 & 2 & | & 4 \\
3 & 2 & | & 2 \\
0 & 4 & | & -2
\end{pmatrix}
\xrightarrow{2 - 3 \cdot 1}
\begin{pmatrix}
1 & 2 & | & 4 \\
0 & -4 & | & -10 \\
0 & 4 & | & -2
\end{pmatrix}
\xrightarrow{3 + 2}
\begin{pmatrix}
1 & 2 & | & 4 \\
0 & -4 & | & -10 \\
0 & 0 & | & -12
\end{pmatrix}
```

Eqn. 3: \( 0x + 0y = -12 \), impossible.

Example 3. Solve the linear system:

\[
\begin{bmatrix}
1 & 2 \\
3 & 2 \\
0 & 4
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
4 \\
2 \\
10
\end{bmatrix}
\]

```
A

\begin{pmatrix}
1 & 2 & | & 4 \\
3 & 2 & | & 2 \\
0 & 4 & | & 10
\end{pmatrix}
\xrightarrow{2 - 3 \cdot 1}
\begin{pmatrix}
1 & 2 & | & 4 \\
0 & -4 & | & -10 \\
0 & 4 & | & 10
\end{pmatrix}
\xrightarrow{3 + 2}
\begin{pmatrix}
1 & 2 & | & 4 \\
0 & -4 & | & -10 \\
0 & 0 & | & 0
\end{pmatrix}
```

Eqn. 2: \(-4y = -10\), \(y = \frac{5}{2}\).

Eqn. 1: \(x + 2(\frac{5}{2}) = 4\), \(x = -1\)

\[
\begin{pmatrix}
x \\
y
\end{pmatrix}
= 
\begin{pmatrix}
\frac{1}{2} \\
\frac{5}{2}
\end{pmatrix}
\]
• Gaussian elimination and pivoting (type 1 + type 2 row operations) can bring any matrix to the following form, which is called row echelon form:

\[
\begin{bmatrix}
\circ & * & \ldots & * & \ldots & * & \ldots & * & \ldots & * & \ldots & * & \ldots & * & \ldots & * \\
0 & 0 & \ldots & 0 & \circ & \ldots & * & \ldots & * & \ldots & * & \ldots & * & \ldots & * \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & \circ & \ldots & * & \ldots & * & \ldots & * & \ldots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & \circ & * & \ldots & * \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \vdots & \vdots & \vdots \\
\end{bmatrix}
\]

• Once in this “staircase” shape, we can solve a linear system with this coefficient matrix from bottom to top, as we just did.

• The circled values are called the pivots.

**Definition:** The number of pivots is called the rank of the matrix \( A \).

\[
\text{rank}(A) = \text{number of its pivots}
\]

Even if a matrix is brought to two different row echelon forms, the ranks are the same.

In other words, the rank depends only on the matrix, not the particular choice of row operations we used to bring it to row echelon form.
Remark:

1. As we’ve seen, any square \((n \times n)\) matrix \(A\) can be brought to upper triangular form, which is a special case of row echelon form.

2. When \(A\) is \textbf{nonsingular}, all the diagonal elements of the upper triangular matrix will be nonzero.

\[
\begin{bmatrix}
A & \rightarrow & U = \begin{bmatrix}
\vdots & \vdots \\
u_{ii} & \hdots \\
\vdots & \vdots \\
u_{nn}
\end{bmatrix} & \text{such that} \quad u_{ii} \neq 0, \quad u_{nn} \neq 0.
\end{bmatrix}
\]

\[A_{n \times n} \text{ is nonsingular} \iff \text{rank}(A_{n \times n}) = n.\]

*Another way of saying this is that nonsingular matrices are “\text{full rank}”, since they have the maximum allowed rank.

Example 4. Example in Lecture 3 again, we have

\[
A = \begin{pmatrix}
1 & 3 & 1 & 2 \\
2 & 6 & 3 & -3 \\
-2 & -6 & -2 & 1 \\
1 & 2 & 1 & 3
\end{pmatrix} \quad \text{type 1+type 2} \quad \text{U = } \begin{pmatrix}
1 & 3 & 1 & 2 \\
0 & -1 & 0 & 1 \\
0 & 0 & 1 & -7 \\
0 & 0 & 0 & 5
\end{pmatrix}.
\]

Then \(\text{rank}(A) = 4\) (full rank).

Some definitions:

- When solving a general linear system, the variables that correspond to columns not containing a pivot can be chosen arbitrarily. These are called \textbf{free variables}.

- The variables corresponding to columns that do contain a pivot are called \textbf{basic variables}.

*We solve for the \textbf{basic} variables in terms of the \textbf{free} variables.
In Example 1, we have seen that

\[
\begin{pmatrix}
1 & 2 & 3 \\
-2 & -2 & -4
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 3 \\
0 & 2 & 2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\]

Free variables: \( z \)

basic variables: \( x, y \)

\[ \text{rank} (A) = 2. \]

**Definition:** We say that a system is **compatible** if the system has at least one solution.

* Note that the compatibility of a system \( A\mathbf{x} = \mathbf{b} \) depends both on the coefficient matrix \( A \) and the right hand side \( \mathbf{b} \).

**Summary:**

- A system may have 0, 1 or infinitely many solutions, but no other numbers.
  *So if there are two solutions, then there must be infinitely many solutions.

- Let \( A \) be a \( m \times n \) matrix. When the system \( A\mathbf{x} = \mathbf{b} \) is compatible and

\[ \text{rank}(A) = \text{number of variables } n, \]

there is **exactly 1 solution**. For instance, see Example. 3 above.

- Let \( A \) be a \( m \times n \) matrix.

<table>
<thead>
<tr>
<th>( m \geq n )</th>
<th>solutions of ( A\mathbf{x} = \mathbf{b} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 1, ( \infty )</td>
<td></td>
</tr>
</tbody>
</table>

- Having a unique (exactly 1) solution is only possible if \( m \geq n \) (i.e. for square or tall coefficient matrices). See Ex 3.
– When \( n > m \) (i.e. the coefficient matrix is short and wide), there are either 0 or infinitely many solutions; see Example 1.

Recall that the rank \( r \) of a matrix is the number of rows that are not identically zero, after the matrix has been brought to row echelon form.

**Fact 1.** Let \( A \) be \( m \times n \) matrix. then

\[
0 \leq r = \text{rank}(A) \leq \min\{m, n\}
\]

§ Homogeneous Systems (\( Ax = 0 \)).

When the right hand side of a linear system is the 0 vector, we say the system is homogeneous. That is, a homogeneous system is of the form \( Ax = 0 \).

- The vector \( x = 0 \) is always a solution to this system, since \( A0 = 0 \).
- If the matrix \( A \) is nonsingular, then \( x = 0 \) is the unique solution of \( Ax = 0 \).

\[
x = A^{-1}b = A^{-1}0 = 0.
\]

**Example 5.** Solve the homogeneous system:

\[
\begin{pmatrix}
1 & 3 & -2 \\
2 & 2 & 4 \\
-1 & -3 & 2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 3 & -2 \\
2 & 2 & 4 \\
-1 & -3 & 2
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 3 & -2 \\
2 & 2 & 4 \\
-1 & -3 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 3 & -2 \\
0 & -4 & 8 \\
0 & 0 & 0
\end{pmatrix}
= 0
\]

Free variable = \( z \)

Basic \( \leftrightarrow \) \( x, y \)

\[ \text{rank } A = 2 \]

\[ x = -4z, \]

\[ y = 2z, \]

\[ x + 3(2z) - 2z = 0 \]

\[ x = -4z. \]
Example 6.

\[
\begin{pmatrix}
1 & 3 & -2 & 1 \\
0 & -2 & -4 & 3 \\
0 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
w \\
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (m < n).
\]

(1) \( z = 0 \).
(2) \(-2x - 4y + 0 = 0 \), \( x = -2y \).
(3) \( w + 3(-2y) - 2y + 0 = 0 \), \( w = 8y \).

\[
\begin{pmatrix}
w \\
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix} 8 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} y, \quad \forall y.
\]

**Fact 2.** Let \( A = A_{m\times n} \) be the matrix of size \( m \times n \).

1. \( Ax = 0 \) has a nontrivial solution \( x \neq 0 \) \( \iff \) \( \text{rank}(A) < n \).
2. If \( m < n \), the system \( Ax = 0 \) always has a nontrivial solution.
3. If \( m = n \), the system \( Ax = 0 \) has a nontrivial solution \( \iff \) \( A \) is singular.

\( A \) is not invertible.

* Later, we will learn that the set of all \( x \) satisfying \( Ax = 0 \) is called the **kernel** of \( A \).