Lecture 7: Quick review from previous lecture

- The transpose of a matrix A^T. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.
- If $A = A^T$, then we call the matrix A is **symmetric** $A = \begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix} = A^T$.
- A is **nonsingular** $\Rightarrow$ $\boxed{PA = LDV}$
- A **symmetric** matrix A is **regular** $\Longleftrightarrow$ $\boxed{A = LDL^T}$

  $\Rightarrow P = I$

  Here P is a permutation matrix, L is lower unitriangular, D is diagonal, and V is upper unitriangular.

A nonsingular,

$$PA = LU = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix} \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}_{V}, \text{ unique.}$$

Today we will discuss

- Sec. 1.8 general system.

- Lecture will be recorded -

1.8 General linear system

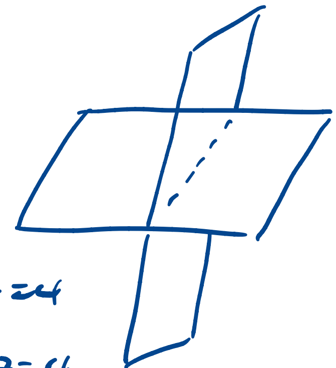
Consider a $m \times n$ matrix A. Here A may be a rectangular matrix or square matrix.

Let's look at the following different situations:

1. **the number of equations < the number of variables** $\boxed{m < n}$:

Example 1. Solve the linear system:

$$\begin{pmatrix} 1 & 2 & 3 \\ -2 & -2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$



$x+2y+3z=4$

$-2x-2y-4z=4$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ -2 & -2 & -4 & 4 \end{array}\right) \xrightarrow{② + 2①} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 12 \end{array}\right)$$

② $2y + 2z = 12$

$y = \dfrac{12-2z}{2} = 6 - z$

① $x + 2(6 - z) + 3z = 4$.

$x = 4 - 3z - (12 - 2z)$

$= -8 - z$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -8-z \\ 6-z \\ z \end{pmatrix}$$ for any scalar z. #.

$\parallel$ infinitely many solutions!

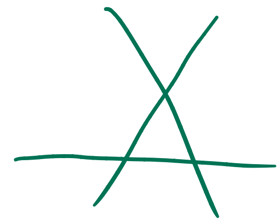
2. the number of equations $\geq$ the number of variables $\boxed{m \geq n}$:

Example 2. Solve the linear system:

$x + 2y = 4$ (line)

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$$

"NO solution"



$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & 2 & 2 \\ 0 & 4 & -2 \end{array}\right) \xrightarrow{②-3①} \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -4 & -10 \\ 0 & 4 & -2 \end{array}\right)$$

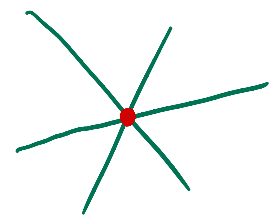
$$\xrightarrow{③+②} \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -4 & -10 \\ 0 & 0 & -12 \end{array}\right)$$

Equ. ③ : $0x + 0y = -12$, impossible.

Example 3. Solve the linear system:

"only one solution"

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 0 & 4 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 10 \end{pmatrix}$$



$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & 2 & 2 \\ 0 & 4 & 10 \end{array}\right) \xrightarrow{②-3①} \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -4 & -10 \\ 0 & 4 & 10 \end{array}\right)$$

$$\xrightarrow{③+②} \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -4 & -10 \\ 0 & 0 & 0 \end{array}\right).$$

- basic variables x, y.
- rank $A = 2$

② : $-4y = -10$, $y = 5/2$.

① : $x + 2\left(\frac{5}{2}\right) = 4$, $\underline{x = -1}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 5/2 \end{pmatrix}$$

§ Row echelon form

- Gaussian elimination and pivoting (type 1 + type 2 row operations) can bring any matrix to the following form, which is called row echelon form:

$$
\begin{pmatrix}
\circledast & * & \dots & * & * & \dots & * & * & \dots & \dots & * & * & * & \dots & * \\
0 & 0 & \dots & 0 & \circledast & \dots & * & * & \dots & \dots & * & * & * & \dots & * \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & \circledast & \dots & \dots & * & * & * & \dots & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & \circledast & * & \dots & * \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0
\end{pmatrix}
$$

- Once in this "**staircase**" shape, we can solve a linear system with this coefficient matrix from bottom to top, as we just did.

- The circled values are called the pivots.

Definition: The number of pivots is called the rank of the matrix A.

$$\boxed{\mathsf{rank}(A) = \text{number of its pivots}}$$

Even if a matrix is brought to two different row echelon forms, **the ranks are the same.**
In other words, the rank depends only on the matrix, not the particular choice of row operations we used to bring it to row echelon form.

Remark:

1. As we've seen, any square ($n \times n$) matrix A can be brought to upper triangular form, which is a special case of row echelon form.
2. When A is **nonsingular**, all the diagonal elements of the upper triangular matrix will be nonzero.

$$A \to U = \begin{bmatrix} u_{11} & & \\ & \ddots & \\ 0 & & u_{nn} \end{bmatrix} \qquad u_{11} \neq 0, \ \vdots, \ u_{nn} \neq 0.$$

$$\boxed{A_{n\times n} \text{ is nonsingular} \iff \mathsf{rank}(A_{n\times n}) = n}$$

*Another way of saying this is that nonsingular matrices are "full rank", since they have the maximum allowed rank.

Example 4. Example in Lecture 3 again, we have

$$A = \underbrace{\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 3 & -3 \\ -2 & -6 & -2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}}_{nonsingular} \underbrace{\longrightarrow}_{type\,1+type\,2} U = \underbrace{\begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 5 \end{pmatrix}}_{\text{row echelon form}}.$$

Then $\mathsf{rank}(A) =$ 4 (full rank).

Some definitions:

- When solving a general linear system, the variables that correspond to columns not containing a pivot can be chosen arbitrarily. These are called **free variables**.
- The variables corresponding to columns that do contain a pivot are called **basic variables**.

*We solve for the **basic** variables in terms of the **free** variables.

In Example 1, we have seen that

$$\overbrace{\begin{pmatrix} 1 & 2 & 3 \\ -2 & -2 & -4 \end{pmatrix}}^{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \end{pmatrix}}_{\text{row echelon form}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

rank(A) = 2.

Free variables: z

basic variables: x, y,

Definition: We say that a system is **compatible** if the system has at least one solution.

* Note that the compatibility of a system $A\mathbf{x} = \mathbf{b}$ depends both on the coefficient matrix A and the right hand side $\mathbf{b}$.

Summary:

- A system may have 0, 1 or infinitely many solutions, but no other numbers. *So if there are two solutions, then there must be infinitely many solutions.
- Let A be a $m \times n$ matrix. When the system $A\mathbf{x} = \mathbf{b}$ is **compatible** and

$$\mathsf{rank}(A) = \text{number of variables n},$$

there is **exactly** 1 **solution**. For instance, see Example. 3 above.

- Let A be a $m \times n$ matrix.

	solutions of $A\mathbf{x} = \mathbf{b}$
$m \geq n$	$0, 1, \infty$
$m < n$	$0, \infty$

– Having a unique (exactly 1) solution is only possible if $\boxed{m \geq n}$ (i.e. for square or tall coefficient matrices). See Ex 3.

– When $n > m$ (i.e. the coefficient matrix is short and wide), there are either 0 or infinitely many solutions; see Example 1.

Recall that the rank r of a matrix is the number of rows that are not identically zero, after the matrix has been brought to row echelon form.

Fact 1. Let A be $m \times n$ matrix. then

$$0 \leq r = \mathsf{rank}(A) \leq min\{m, n\}$$

homogeneous system.

§ Homogeneous Systems ($A\mathbf{x} = \mathbf{0}$).

When the right hand side of a linear system is the $\mathbf{0}$ vector, we say the system is homogeneous. That is, a homogeneous system is of the form $\boxed{A\mathbf{x} = \mathbf{0}}$.

- The vector $\mathbf{x} = \mathbf{0}$ is always a solution to this system, since $A\mathbf{0} = \mathbf{0}$.
- If the matrix A is nonsingular, then $\mathbf{x} = \mathbf{0}$ is the unique solution of $A\mathbf{x} = \mathbf{0}$.

$$\underline{x = A^{-1}b} = A^{-1}0 = 0.$$

Example 5. Solve the homogeneous system:

$$\overbrace{\begin{pmatrix} 1 & 3 & -2 \\ 2 & 2 & 4 \\ -1 & -3 & 2 \end{pmatrix}}^{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Free variable = z

basic " = x, y

rank A = $\underline{2}$

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 2 & 2 & 4 & 0 \\ -1 & -3 & 2 & 0 \end{array}\right) \xrightarrow[\text{③}+\text{①}]{\text{②}-2\text{①}} \left(\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

② : $-4y + 8z = 0.$

$y = 2z.$

① : $x + 3(2z) - 2z = 0$

$x = -4z.$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix} z,$$

for all z.

$(\forall z)$

.

rank A : 3

free variables : y

basic " : w, x, z.

Example 6.

$$\overbrace{\begin{pmatrix} 1 & 3 & -2 & 1 \\ 0 & -2 & -4 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}}^{A_{m\times n}} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$(m < n)$.

③ $z = 0$.

② $-2x - 4y + 0 = 0$.

$x = -2y$.

① $w + 3(-2y) - 2y + 0 = 0$

$w = 8y$.

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \\ 1 \\ 0 \end{pmatrix} y, \quad \forall y.$$

Fact 2. Let $A = A_{m\times n}$ be the matrix of size $m \times n$.

(1)
$$\boxed{A\mathbf{x} = \mathbf{0} \text{ has a } \overbrace{\text{nontrivial}}^{x \neq 0} \text{ solution } \mathbf{x} \neq \mathbf{0} \iff \operatorname{rank}(A) < n}$$

(2) If $m < n$, the system $A\mathbf{x} = \mathbf{0}$ always has a nontrivial solution.

(3) If $m = n$, the system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\iff$ A is singular.
(A is NOT invertible).

* Later, we will learn that the set of all $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{0}$ is called the **kernel** of A.