Lecture 8: Quick review from previous lecture

- Gaussian elimination (with pivoting) can bring any matrix to the following form, which is called row echelon form:

$$
\begin{pmatrix}
\circledast & * & \cdots & * & * & \cdots & * & * & \cdots & \cdots & * & * & * & \cdots & * \\
0 & 0 & \cdots & 0 & \circledast & \cdots & * & * & \cdots & \cdots & * & * & * & \cdots & * \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \circledast & \cdots & \cdots & * & * & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \circledast & * & \cdots & * \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
$$

- The number of pivots is called the rank of the matrix A.

$$\mathsf{rank}(A) = \text{number of its pivots}$$

- $n \times n$ matrix A is nonsingular if and only if $\mathsf{rank}(A) = n$. (full rank)

- This system $A\mathbf{x} = \mathbf{0}$ is called homogeneous.

  x=0 is always a solution

Today we will discuss

- Sec. 1.9 the determinant and Sec. 2.1 vector space

- Lecture will be recorded -

- HW 2 is due Today by 6 pm.

1.9 Determinants

Recall that we saw previously that a 2-by-2 matrix

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible $\Leftrightarrow$ $\mathsf{det}(A) = ad - bc \neq 0$ $\Leftrightarrow$ A is nonsingular

Today we will now see how to generalize this to all square matrices. The key ingredient will be the permuted LU factorization that we have already seen.

§ Generalize to $n \times n$ matrix.

Indeed, Gaussian elimination with pivoting can turn

$$\textbf{Any} \text{ square matrix } A \Longrightarrow \boxed{PA = LU}$$

where P is a permutation matrix, L is lower unitriangular, and U is upper triangular. **This matrix U here could have "zero" diagonal entries.*

- We have known that A is invertible precisely (by Gauss-Jordan elimination) when $A \longrightarrow U = \begin{pmatrix} u_{11} & \cdots & & \\ 0 & u_{22} & & \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & u_{nn} \end{pmatrix}$ with all $u_{11}, \ldots, u_{nn}$ are nonzero.
- Thus, we conclude that the *product* of all these numbers,

$$\boxed{\mathsf{det}(U) = u_{11} \cdots u_{nn} \neq 0 \iff A \text{ is invertible (nonsingular)}}$$

Definition: Motivated by this, we **define** $\mathsf{det}(A)$ as follows:

$$\mathsf{det}(A) = (-1)^k \mathsf{det}(U) = (-1)^k \prod_{i=1}^{n} u_{i,i}$$

where k denotes the number of row permutations we performed to bring A into upper triangular form.

Fact 1: We have

$$\det(A) \neq 0 \iff A \text{ is nonsingular (invertible)}$$

Example 1. Compute the determinant of

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & -3 & 4 \\ 0 & 2 & -2 & 3 \\ 1 & 1 & -4 & -2 \end{pmatrix}$$

$$\xrightarrow[④-①]{②-2①} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 3 \\ 0 & 1 & -3 & -4 \end{pmatrix} \xrightarrow[④-②]{③-2②} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -2 & -4 \end{pmatrix}$$

$$\xrightarrow[③\ ④]{\text{permute}} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 3 \end{pmatrix} = U.$$

$$\det A = (-1)^1\, 1 \cdot 1 \cdot (-2) \cdot 3 = 6 \ \#.$$

A is nonsingular, invertible.

An immediate result:

Note that if $A_{n\times n}$ has a row consisting entirely of zeros, then $\mathsf{det}(A) = 0$.

Example 2. Compute the determinant of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 0 & 4 & 6 \end{pmatrix}$$

$$A \xrightarrow{②-3①} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 4 & 6 \end{pmatrix} \xrightarrow{② \leftrightarrow ③} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\det A = (-1)^1 \, 1 \cdot 4 \cdot 0 = 0.$$

2nd row = 3 (1st row).

§ Elementary Row Operations on the determinant of a $n \times n$ matrix A:

If B is a matrix obtained by adding a multiple of one row of A to another row of A. Then

$$\mathsf{det}(A) = \mathsf{det}(B).$$

Example 3.

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \quad B = \underbrace{\begin{pmatrix} a+5d & b+5e & c+5f \\ d & e & f \\ g & h & i \end{pmatrix}}_{row\,1\,of\,A+5(row\,2\,of\,A)}$$

$\to U \qquad\qquad \to U$

If B is a matrix obtained by interchanging any two rows of A once, then

$$\det(B) = -\det(A).$$

Example 4.

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \quad B = \underbrace{\begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}}_{switch\,row\,1\,and\,row\,2\,of\,A}$$

If B is a matrix obtained by multiplying a row of A by a nonzero scalar k, then

$$\det(B) = k\det(A).$$

Example 5. $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{2\,②} B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 10 & 14 \\ 0 & 0 & 1 \end{pmatrix}$

$\det A = 1 \cdot 5 \cdot 1$, $\det B = 1 \cdot \underline{\underline{10}} \cdot 1$

$= 1 \cdot 2(5) \cdot 1$

$\det B = 2 \det A$.

Then for scalar c, we can derive

$A_{n\times n}$. $\boxed{\det(cA) = c^n \det(A)}$

$\tilde{B} = 2A = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 10 & 14 \\ 0 & 0 & 2 \end{pmatrix}$.

$\det \tilde{B} = 2^3 \det A$.

§ **The det operator behaves well with matrix multiplication, inversion, and transposition (but NOT addition!).**

(1) If A and B are two square matrices of the same size, then

$$\boxed{\det(AB) = \det(A)\det(B)}$$

$$= \det(BA).$$

*In particular, note that $\det(AB) = \det(BA)$, even though AB need not equal BA.

$AB \neq BA$.

(2) AB is invertible $\Leftrightarrow$ both A and B are invertible.

$\det(A+B) \neq \det A + \det B$.

EX: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

(3) If A is invertible, then

$$\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

$\det(A+B) = 0$

$\det A = 1 = \det B$

NOT the same

$\det A + \det B = 2$.

(4) $\boxed{\det(A^T) = \det(A)}$

[To see this]

(2) AB is invertible $\overset{\text{Fact 1}}{\Longleftrightarrow} \det(AB) \neq 0$.

$\overset{(1)}{\Longleftrightarrow} \det A \det B \neq 0$

$\Longleftrightarrow \det A \neq 0,\ \det B \neq 0.$

$\overset{\text{Fact 1}}{\Longleftrightarrow} A, B$ is invertible

(3) $\det(I) = \det(AA^{-1}) \overset{(1)}{=} \det A \det A^{-1}$

$1 =$

$\det(A^{-1}) = \dfrac{1}{\det A}.$

§ There is a formula for $\det(A)$.

Given a $n \times n$ matrix A, we denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j by $\tilde{A}_{ij}$.

Example 6. $A = \begin{pmatrix} 1 & 3 & 0 \\ -2 & -5 & -3 \\ 1 & 2 & 2 \end{pmatrix}$ we have

$$A = \begin{pmatrix} 1 & 3 & 0 \\ -2 & -5 & -3 \\ 1 & 2 & 2 \end{pmatrix}$$

$$\tilde{A}_{11} = \begin{pmatrix} -5 & -3 \\ 2 & 2 \end{pmatrix}, \quad \tilde{A}_{12} = \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix}, \quad \tilde{A}_{23} = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$$

1st row 2nd column

Let A be $n \times n$ matrix.

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \cdot \det(\tilde{A}_{ij})$$

Example 7. Compute the determinant of the following A:

$$A = \begin{array}{ccc} a_{11} & a_{12} & a_{13} \end{array}$$
$$A = \begin{pmatrix} 1 & 3 & 0 \\ -2 & -5 & -3 \\ 1 & 2 & 2 \end{pmatrix}$$

$$\det A = a_{11} \det \tilde{A}_{11} + \underbrace{(-1)^{1+2}}_{=-1} a_{12} \det \tilde{A}_{12} + a_{13} \det \tilde{A}_{13}$$

$$= 1 \det \begin{pmatrix} -5 & -3 \\ 2 & 2 \end{pmatrix} - 1\,(3) \det \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix} + 0 \det \tilde{A}_{13}$$

$$= -5 \cdot 2 - (2)(-3) - 1\;3\,(-4+3) + 0.$$

$$= -1 \quad \#$$

EX: $A_{2\times2}$, $B_{2\times2}$. $\det A = 2$, $\det B = 3$.

Find $\det(7AB^2) = 7^2 \det(AB^2)$

$= 7^2 \det A\,(\det B)^2$