Lecture 9: Quick review from previous lecture

- If $A \longrightarrow U = \begin{pmatrix} u_{11} & & & \\ 0 & u_{22} & & \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & u_{nn} \end{pmatrix}$, then

$$\boxed{\mathsf{det}(A) = (-1)^k \prod_{i=1}^{n} u_{ii}}$$

  where k denotes the number of row permutations we performed to bring A into upper triangular form.

- Let A, B are $n \times n$ matrices.

$\det(A+B) \neq \det A + \det B$.

$$\boxed{\mathsf{det}(cA) = c^n \mathsf{det}(A), \qquad \mathsf{det}(AB) = \mathsf{det}(A)\mathsf{det}(B)}$$

- If A is invertible, then

$$\boxed{\mathsf{det}(A^{-1}) = \frac{1}{\mathsf{det}(A)}}$$

- $$\boxed{\mathsf{det}(A) \neq 0 \iff A \text{ is nonsingular}}$$

Today we will discuss

- Sec. 2.1 vector space

- Lecture will be recorded -

2.1 Real Vector Spaces

Definition: A vector space is a set V equipped with two operations:

(1) (Addition) If $\mathbf{v}, \mathbf{w} \in V$, then $\mathbf{v} + \mathbf{w} \in V$.

(2) (Scalar Multiplication) Multiplying a vector $\mathbf{v} \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $c\mathbf{v} \in V$.

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars $c, d \in \mathbb{R}$:

(a) *Commutativity of Addition*: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
(b) *Associativity of Addition*: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
(c) *Additive Identity*: There is a zero element $\mathbf{0} \in V$ satisfying $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$.
(d) *Additive Inverse*: For each $\mathbf{v} \in V$ there is an element $-\mathbf{v} \in V$ such that
$\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = (-\mathbf{v}) + \mathbf{v}$.
(e) *Distributivity*: $(c + d)\mathbf{v} = (c\mathbf{v}) + (d\mathbf{v})$, and $c(\mathbf{v} + \mathbf{w}) = (c\mathbf{v}) + (c\mathbf{w})$.
(f) *Associativity of Scalar Multiplication*: $c(d\mathbf{v}) = (cd)\mathbf{v}$.
(g) *Unit for Scalar Multiplication*: the scalar $1 \in \mathbb{R}$ satisfies $1\mathbf{v} = \mathbf{v}$.

Example 1. $\mathbb{R}^n$ is a vector space with addition and scalar multiplication defined by

$$\boxed{\mathbf{u} + \mathbf{v} = (a_1 + b_1, \ldots, a_n + b_n)^T, \qquad c\mathbf{u} = (ca_1, \ldots, ca_n)^T}$$

where $\mathbf{u} = (a_1, \ldots, a_n)^T$ and $\mathbf{v} = (b_1, \ldots, b_n)^T$.

(1) (2) is okay.

Check (a) - (g):

(a) $u + v = v + u$.

(b) ...

(c) zero element is $\vec{0} = (0, \ldots, 0)$ in $\mathbb{R}^n$. $\vec{0} + v = v = v + \vec{0}$.

(d) $v + (-v) = (a_1, \ldots, a_n) + (-a_1, \ldots, -a_n) = \vec{0}$.

(e)
(f) $\Big\}$ fine.
(g)

Example 2. We denote the set of all $m \times n$ matrices with entries from $\mathbb{R}$ by

$$\mathsf{M}_{m\times n}(\mathbb{R}) := \{A : \ A \text{ is } m \times n \text{ matrix}\}$$

with the following operations of **matrix addition** and **scalar multiplication**:

For $A, B \in \mathsf{M}_{m\times n}(\mathbb{R})$ and $c \in \mathbb{R}$,

$$(A+B)_{\boxed{ij}} = A_{ij} + B_{ij}, \qquad (cA)_{ij} = cA_{ij}$$

i-row, j column

Then $\mathsf{M}_{m\times n}(\mathbb{R})$ is a vector space. Ex: $M_{2\times 3} = \left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23}\end{pmatrix}\right\}$ is a vector space.

Check: (1) (2) okay.

Check: (a) – (g).

(c): zero element is zero matrix $O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. $O + A = A$.

(d): $A + (-A) = O$.

(e) – (g) = ok.

Example 3. Consider the space

$$\mathcal{P}^{(n)} = \{p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0\}$$

consisting of all real polynomials of degree $\leq n$. It has the following operations of **matrix addition** and **scalar multiplication**:

For $p(x) = a_n x^n + \cdots + a_1 x + a_0$ and $q(x) = b_n x^n + \cdots + b_1 x + b_0$,

$$(p+q)(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_1 + b_1)x + (a_0 + b_0)$$
$$cp(x) = ca_n x^n + ca_{n-1}x^{n-1} + \cdots + ca_1 x + ca_0$$

(1) (2) Okay.

(a) (b)

(c) zero element is zero polynomial 0 ($a_n = 0, \ldots, a_0 = 0$).

$0 + p(x) = p(x)$.

(d) $p + (-p) = 0$ (zero polynomial).

(e) – (g).

$S = \{a_n x^n + \cdots + a_0 ;\ a_n \neq 0\}$.

Example 4. Consider the set S of polynomials of degree equal to n with same addition and scalar multiplication as in Example 3. Is such S a vector space? (NO)

(1) $x^n + (-x^n) = 0 \notin S$ (2). $cp \notin S$, if $c \neq 0$.

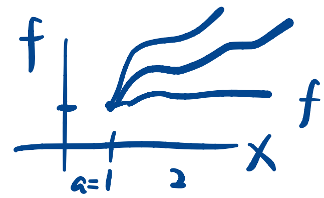
(a)

⋮ zero polynomial is NOT in S.

(g).

Notation: $\notin$ (NOT in)

$\in$ (in)



Example 5. We consider the set $S = \{f \in \mathcal{F}([a,b]) : \ f(a) = 1\}$. Here $\mathcal{F}([a,b])$ is the collection of all functions f defined on an interval $[a,b]$.

For $f, g \in S$ and a scalar c,

$$(f+g)(x) \overset{①}{=} f(x) + g(x), \qquad (c \cdot f)(x) = c \cdot f(x)$$

Is S a vector space? (NO).

(1) $f, g \in S$. $f(a) = 1$, $g(a) = 1$.

$(f+g)(a) \overset{①}{=} f(a) + g(a) = 1 + 1 = 2$.

Then $f+g$ is NOT in S.

Q: Is $S' = \{f \in \mathcal{F}([a,b]) : \ f(a) = 0\}$ with same addition and scalar multiplication as above a vector space? (Yes). Exercise.

Example 6. Let $S = \{(a_1, a_2) : \ a_1, a_2 \in \mathbb{R}\} = \mathbb{R}^2$

We define the addition and scalar multiplication by

$$(a_1, a_2) + (b_1, b_2) \overset{1}{=} (a_1 + b_1, a_2 - b_2), \qquad c(a_1, a_2) = (ca_1, ca_2)$$

Is S a vector space? (NO)

(1) $(a_1, a_2) + (b_1, b_2) \overset{1}{=} (a_1 + b_1, a_2 - b_2)$ in S.

(2) $c(a_1, a_2) \overset{2}{=} (ca_1, ca_2)$ in S.

(a) u + v = v + u: $(1,2) + (1,7) \overset{1}{=} (1+1, 2-7)$ $\neq$
$(1,7) + (1,2) \overset{1}{=} (1+1, 7-2)$

Example 7. We let V be the upper right quadrant of $\mathbb{R}^2$, i.e.

$$V = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}.$$

We define addition of vectors and scalar multiplication by:

$$(x, y) \oplus (w, z) \overset{1}{=} (xw, yz), \qquad c \odot (x, y) = (x^c, y^c)$$

* Here we use these notations $\oplus, \odot$ to distinguish it from ordinary ones.

Is V a vector space? (Yes)

(1) $(x, y) \oplus (w, z) \overset{1}{=} (xw, yz) \in V$ positive

(2) $c \odot (x, y) = (x^c, y^c) \in V$ since $x^c, y^c > 0$

(a) ok, (b) ok.

(c) zero element: let zero element $(a, b) = (1, 1)$.

$(a, b) \oplus (x, y) = (x, y)$

$\overset{1}{=} (ax, by)$ $\Longrightarrow$ $a = 1$, $b = 1$

(d) additive inverse: $v = (x, y)$. Let $-v = (\frac{1}{x}, \frac{1}{y})$

$v \oplus (-v) = (x, y) \oplus (\frac{1}{x}, \frac{1}{y}) \overset{1}{=} (1, 1)$ (zero element)

Check (e)–(g)