Lecture 9: Quick review from previous lecture

• If \( A \rightarrow U = \begin{pmatrix} u_{11} & 0 & \cdots & 0 \\ 0 & u_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{nn} \end{pmatrix} \), then

\[
\det(A) = (-1)^k \prod_{i=1}^{n} u_{ii}
\]

where \( k \) denotes the number of row permutations we performed to bring \( A \) into upper triangular form.

• Let \( A, B \) are \( n \times n \) matrices.

\[
\det(cA) = c^n \det(A), \quad \det(AB) = \det(A)\det(B)
\]

• If \( A \) is invertible, then

\[
\det(A^{-1}) = \frac{1}{\det(A)}
\]

• \( \det(A) \neq 0 \iff A \) is nonsingular

Today we will discuss

• Sec. 2.1 vector space

- Lecture will be recorded -

MATH 4242-Week 4-1  
Spring 2021  
1
2.1 Real Vector Spaces

**Definition:** A vector space is a set $V$ equipped with two operations:

1. **Addition** If $v, w \in V$, then $v + w \in V$.
2. **Scalar Multiplication** Multiplying a vector $v \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $cv \in V$.

For all $u, v, w \in V$ and all scalars $c, d \in \mathbb{R}$:

(a) **Commutativity of Addition:** $v + w = w + v$.
(b) **Associativity of Addition:** $u + (v + w) = (u + v) + w$.
(c) **Additive Identity:** There is a zero element $0 \in V$ satisfying $v + 0 = v = 0 + v$.
(d) **Additive Inverse:** For each $v \in V$ there is an element $-v \in V$ such that $v + (-v) = 0 = (-v) + v$.
(e) **Distributivity:** $(c + d)v = (cv) + (dv)$, and $c(v + w) = (cv) + (cw)$.
(f) **Associativity of Scalar Multiplication:** $c(dv) = (cd)v$.
(g) **Unit for Scalar Multiplication:** the scalar $1 \in \mathbb{R}$ satisfies $1v = v$.

**Example 1.** $\mathbb{R}^n$ is a vector space with addition and scalar multiplication defined by

$$u + v = (a_1 + b_1, \ldots, a_n + b_n)^T, \quad cu = (ca_1, \ldots, ca_n)^T$$

where $u = (a_1, \ldots, a_n)^T$ and $v = (b_1, \ldots, b_n)^T$.

(1) (2) is okay.

**Check (a) - (g):**

(a) $u + v = v + u$.
(b) ...
(c) Zero element is $0 = (0, \ldots, 0) \in \mathbb{R}^n$. $0 + v = v$.
(d) $v + (-v) = (a_1, \ldots, a_n) + (-a_1, \ldots, -a_n) = 0$.
(e) ...
(f) ...
(g) ...
Example 2. We denote the set of all $m \times n$ matrices with entries from $\mathbb{R}$ by

$$\mathbb{M}_{m \times n}(\mathbb{R}) := \{ A : A \text{ is } m \times n \text{ matrix} \}$$

with the following operations of **matrix addition** and **scalar multiplication**:

For $A, B \in \mathbb{M}_{m \times n}(\mathbb{R})$ and $c \in \mathbb{R}$,

$$(A + B)_{ij} = A_{ij} + B_{ij}, \quad (cA)_{ij} = cA_{ij}$$

Then $\mathbb{M}_{m \times n}(\mathbb{R})$ is a **vector space**.

**EX:** $\mathbb{M}_{2 \times 3} = \{(a_{11}, a_{12}, a_{13}) \}$ is a vector space.

Check: (1) (2) okay.

Check: (a) (b).

(c): zero element is zero matrix $O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $O + A = A$.

(d): $A + (-A) = O$.

(e): $-(O) = O$.

Example 3. Consider the space

$$\mathcal{P}(n) = \{ p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \}$$

consisting of all real polynomials of degree $\leq n$. It has the following operations of **matrix addition** and **scalar multiplication**:

For $p(x) = a_n x^n + \cdots + a_1 x + a_0$ and $q(x) = b_n x^n + \cdots + b_1 x + b_0$,

$$(p + q)(x) = (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \cdots + (a_1 + b_1) x + (a_0 + b_0)$$

$$cp(x) = c a_n x^n + c a_{n-1} x^{n-1} + \cdots + c a_1 x + c a_0$$

(1) (2) okay.

(a) (b)

(c) zero element is zero polynomial $O (a_n = 0, \ldots, a_0 = 0)$.

$$0 + p(x) = p(x)$$

(d) $p + (-p) = 0$ (zero polynomial).

(e) $-(p) = p$. 

MATH 4242 - Week 4-1

Spring 2021
Example 4. Consider the set $S$ of polynomials of degree equal to $n$ with same addition and scalar multiplication as in Example 3. Is such $S$ a vector space?

\[ S = \{ a_n x^n + \cdots + a_0 : a_n \neq 0 \} \]

1. $x^n + (-x^n) = 0 \notin S$.

2. $c p \in S$ if $c \neq 0$.

(a) Zero polynomial is not in $S$.

Example 5. We consider the set $S = \{ f \in \mathcal{F}([a, b]) : f(a) = 1 \}$.

Here $\mathcal{F}([a, b])$ is the collection of all functions $f$ defined on an interval $[a, b]$.

For $f, g \in S$ and a scalar $c$,

\[
(f + g)(x) = f(x) + g(x), \quad (c \cdot f)(x) = c \cdot f(x)
\]

Is $S$ a vector space? (NO)

1. $f, g \in S$. $f(a) = 1$, $g(a) = 1$.

\[
(f + g)(a) = f(a) + g(a) = 1 + 1 = 2.
\]

Then $f + g$ is not in $S$.

Q: Is $S' = \{ f \in \mathcal{F}([a, b]) : f(a) = 0 \}$ with same addition and scalar multiplication as above a vector space? (Yes) Exercise.
Example 6. Let \( S = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\} = \mathbb{R}^2 \)

We define the addition and scalar multiplication by

\[
(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2), \quad c(a_1, a_2) = (ca_1, ca_2)
\]

Is \( S \) a vector space?

\[
\begin{align*}
1) \quad (a_1, a_2) + (b_1, b_2) &\in S, \\
2) \quad c(a_1, a_2) &\in S, \\
(a) \quad u + v &\in V + u: (1, 2) + (1, 1) = (2, 3) \neq (1 + 1, 2 - 7) \\
(b) \quad (1, 7) + (1, 2) &\in (1 + 1, 7 - 2)
\end{align*}
\]

Example 7. We let \( V \) be the upper right quadrant of \( \mathbb{R}^2 \), i.e.

\( V = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\} \).

We define addition of vectors and scalar multiplication by:

\[
(x, y) \oplus (w, z) = (xw, yz), \quad c \odot (x, y) = (x^c, y^c)
\]

* Here we use these notations \( \oplus, \odot \) to distinguish it from ordinary ones.

Is \( V \) a vector space?

\[
\begin{align*}
1) \quad (x, y) \oplus (w, z) &\in V, \\
2) \quad c \odot (x, y) &\in V \text{ since } x^c, y^c > 0, \\
(a) \quad \text{ok, } \quad (b) \quad \text{ok.}
\end{align*}
\]

(c) **zero element:** let zero element \((a, b) = (1, 1)\).

\[
(a, b) \oplus (x, y) = (x, y) \quad \Rightarrow \quad a = 1, \quad b = 1
\]

(d) **additive inverse:** \( v = (x, y) \). Let \( -v = (\frac{1}{x}, \frac{1}{y}) \)

\[
(\check{(e)-(g)})
\]