In the sequel, $S_n(f)$ refers to the Fourier partial sum as dealt with in class. For $f$ a function defined on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$,

$$S_n(f) = \sum_{|k| \leq n} \hat{f}(k)e_k, \quad e_k = \frac{1}{\sqrt{2\pi}} \exp(ikx),$$

where

$$\langle f, g \rangle = \int_{\mathbb{T}} f \overline{g}dx, \quad \hat{f}(k) = \langle f, e_k \rangle, \quad \text{and} \quad \|f\|_2 = \sqrt{\langle f, f \rangle}.$$

1. Show the following.

   (a) For $f \in C(\mathbb{T})$, show that

   $$(S_n)^2(f) = S_n(S_n(f)) = S_n(f).$$

   (b) For $f, g \in C(\mathbb{T})$, show that

   $$\langle S_n(f), g \rangle = \langle f, S_n(g) \rangle.$$

   (c) Use the above two results and the Cauchy-Schwartz inequality to show that:

   $$\|S_n(f)\|_2 \leq \|f\|_2.$$

2. A function $f \in C(\mathbb{T})$ is said to be Hölder continuous of order $\alpha$ if:

   $$|f(x) - f(y)| \leq C |x - y|^\alpha, \quad 0 < \alpha < 1, x, y \in \mathbb{T}.$$  \hspace{1cm} (1)

   for some constant $C$ that does not depend on $x$ or $y$. Note that Hölder continuity is a kind of $\alpha$-differentiability.

   (a) If $f \in C(\mathbb{T})$, show

   $$\hat{f}'(k) = -\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x + \pi/k)e^{-ikx}dx.$$

   Furthermore, show

   $$\hat{f}'(k) = \frac{1}{2\sqrt{2\pi}} \int_0^{2\pi} [f(x) - f(x + \pi/k)]e^{-ikx}dx.$$
(b) If $f$ is Hölder continuous of order $\alpha$ for $0 < \alpha < 1$, then use (a) to show

$$\hat{f}(k) = O\left(\frac{1}{|k|^\alpha}\right).$$

(c) If $f$ is Hölder continuous where $\alpha > 1/2$, show that $S_n f(x)$ converges to $f(x)$ at every point in $\mathbb{T}$. (Hint: Use the argument in the proof of Theorem 1.)

3. Suppose that $f \in C^1(\mathbb{T})$. Show that

$$\int_0^{2\pi} |f|^2 dx = \sum_{k=-\infty}^{\infty} |\langle f, e_k \rangle|^2.$$

4. Suppose $f \in C^1(\mathbb{T})$ is a real-valued function, and

$$\int_{\mathbb{T}} f \, dx = 0.$$

Show that

$$\int_{\mathbb{T}} f^2 \, dx \leq \int_{\mathbb{T}} f'^2 \, dx.$$

When does the equality hold? This is an instance of the Poincaré inequality.

5. Consider the following convolution operation:

$$f \ast g \equiv \int_{-\pi}^{\pi} f(y)g(x-y) \, dy$$

for $f, g \in C(\mathbb{T})$.

(a) Show that

$$\hat{f} \ast g(k) = \sqrt{2\pi} \hat{f}(k) \hat{g}(k)$$

(b) Suppose $f \in C(\mathbb{T})$ and $g \in C^\infty(\mathbb{T})$. Use the above result to argue that $f \ast g$ should also be $C^\infty(\mathbb{T})$. 

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