MATH 8401, Homework 1
(Due Wed. in class, Sep. 19, 2018)

1. Problems from Handout:
   (a) Problem from Section 1.1: 5, 6.
   (b) Problem from Section 1.2: 3, 9, 10. (Note that the notation Λ in 10(c) means a diagonal matrix.)
   (c) Problem from Section 1.3: 4.

2. In this problem, we consider a matrix A such that $AA^* = A^*A$, where A^* is the adjoint matrix (conjugate transpose). Such a matrix is called a *normal matrix*. In addition, we say a space S is invariant under a matrix T if for any $x \in S$ such that $Tx \in S$.
   (a) $\|Ax\| = \|A^*(x)\|$ for all x where $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$.
   (b) $A - cI$ is normal for every constant c.
   (c) Show that $N(A) = N(A^*)$ and $R(A)^\perp = R(A^*)^\perp$.
   (d) If λ_1 and λ_2 are distinct eigenvalues of A with corresponding eigenvectors x_1 and x_2, then x_1 and x_2 are orthogonal.
   (e) Suppose there are two square matrices A and B that commute: $AB = BA$. Then, show that there is at least one eigenvector that is common to the two matrices. (*Hint*: Show any eigenspace of A is invariant under B.)
   (f) Let A be square matrix acting on a vector space $\mathcal{V}$ and suppose $\mathcal{W} \subset \mathcal{V}$ is a subspace of $\mathcal{V}$. Suppose $\mathcal{W}$ is invariant under A. Show that $\mathcal{W}^\perp$ is also invariant under A^*.
   (g) Show that normal matrices can be diagonalized using a unitary matrix.

3. We define a matrix is *positive definite* (*positive semidefinite*) if A is self-adjoint and $\langle Ax, x \rangle > 0$ ($\langle Ax, x \rangle \geq 0$) for all $x \neq 0$. Suppose that $A = A^*$ and $B = B^*$. Prove the following results.
   (a) A is positive definite (semidefinite) if and only if all eigenvalues of A are positive (nonnegative).
   (b) Suppose A is positive semidefinite if and only if $A = B^*B$ for some square matrix B.

(c) If A and B are positive semidefinite such that $A^2 = B^2$, then $A = B$.

4. Let A be an $n \times n$ real symmetric matrix. Let H be an $n - j$ dimensional subspace of $\mathbb{R}^n$, where $n - j \geq 1$. Let $\boldsymbol{q}_1, \cdots, \boldsymbol{q}_{n-j}$ be a set of orthonormal vectors spanning the subspace H. Form the $n \times (n - j)$ matrix with columns $\boldsymbol{q}_1, \cdots, \boldsymbol{q}_{n-j}$ and call this matrix Q. Define:

$$A_H = Q^T A Q.$$

Le the eigenvalues of A be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and let the eigenvalues of A_H be $\widehat{\lambda}_1 \geq \widehat{\lambda}_2 \geq \cdots \geq \widehat{\lambda}_{n-j}$.

(a) Show that the eigenvalues of A_H do not depend on the choice of orthonormal basis on H.

(b) Let S_{k-1} be the set of all $n - k + 1$ dimensional subspaces of $\mathbb{R}^n$, and $\widehat{S}_{k-1}$ be the set of all $(n - j) - (k - 1)$ dimensional subspaces of H (assuming $(n - j) - (k - 1) \geq 1$). Show that:

$$\begin{aligned} &\min_{P \in S_{k-1}} \max_{\|\boldsymbol{x}\|=1, \boldsymbol{x} \in P} \langle A\boldsymbol{x}, \boldsymbol{x} \rangle \geq \min_{P \in S_{k-1}} \max_{\|\boldsymbol{x}\|=1, \boldsymbol{x} \in P \cap H} \langle A\boldsymbol{x}, \boldsymbol{x} \rangle \\ \geq &\min_{P \in \widehat{S}_{k-1}} \max_{\|\boldsymbol{x}\|=1, \boldsymbol{x} \in P} \langle A\boldsymbol{x}, \boldsymbol{x} \rangle \geq \min_{P \in S_{k+j-1}} \max_{\|\boldsymbol{x}\|=1, \boldsymbol{x} \in P} \langle A\boldsymbol{x}, \boldsymbol{x} \rangle . \end{aligned}$$

(c) Use the above to show that:

$$\lambda_k \geq \widehat{\lambda}_k \geq \lambda_{k+j}, \; 1 \leq k \leq n - j.$$

This is the content of Theorem 1.8 in handout, with essentially the same proof.