Math 8583: Theory of Partial Differential Equations: Fall 2013

Homework #1 (due on Friday, September 27, till 4:30 pm)

50 points are distributed between 5 problems, 10 points each.

1. Show that Laplace’s equation is invariant with respect to orthogonal transformations: if $u \in C^2(\Omega)$ satisfies $\Delta u(x) \equiv 0$ in $\Omega$, then $v(y) := u(Ay)$ satisfies $\Delta v(y) \equiv 0$ in the domain \( \{ y \in \mathbb{R}^n : Ay \in \Omega \} \). Here $A$ is any orthogonal matrix, i.e. its transposed matrix $A^T = A^{-1}$.

2. Let $f = f(x)$ be a continuous function on $[-1, 1]$. Show that the problem
\[
    u'' + f = 0 \quad \text{in} \quad (-1, 1), \quad u(\pm 1) = 0,
\]
has a unique solution $u \in C^2([-1, 1])$, which has the form $u(x) = \int_{-1}^{1} G(x, y) f(y) dy$, $-1 \leq x \leq 1$, where the function $G(x, y)$ (Green’s function for this problem) does not depend on $f$. Find the explicit expression of $G(x, y)$ for $x, y \in [-1, 1]$.

3. For a given function $f \in C([-1, 1])$, let $u \in C^2([-1, 1])$ be a solution to the problem (1). Show that for arbitrary constant $\gamma \in (0, 1)$, there exists a constant $N$ depending only on $\gamma$, such that
\[
    \sup_{(-1,1)} d^{-\gamma}|u| \leq N \cdot \sup_{(-1,1)} d^{2-\gamma}|f|, \quad \text{where} \quad d = d(x) := 1 - |x|.
\]
Show that the above estimate fails for $\gamma = 0$.

Hint. One can either use the previous problem, or the comparison function $v = A_1 d^\gamma \zeta + A_2 (1 - x^2)$ with some positive constants $A_1, A_2$ and a smooth function $\zeta$ satisfying
\[
    0 \leq \zeta \leq 1 \quad \text{on} \quad \mathbb{R}^1, \quad \zeta(x) \equiv 0 \quad \text{for} \quad |x| \leq \frac{1}{3}, \quad \zeta(x) \equiv 1 \quad \text{for} \quad |x| \geq \frac{2}{3}.
\]

4. Let $K = \text{const} \geq 0$. Show that the problem
\[
    U'' + K |U'| + 1 = 0 \quad \text{in} \quad (-1, 1), \quad U(\pm 1) = 0,
\]
has a unique solution $U \in C^2([-1, 1])$. Moreover, show that any solution $u \in C^2([-1, 1])$ to the problem
\[
    u'' + pu' + 1 = 0 \quad \text{in} \quad (-1, 1), \quad u(\pm 1) = 0, \quad \text{where} \quad |p(x)| \leq K,
\]
satisfies the estimate $0 \leq u \leq U$ in $(-1, 1)$.

5. Let $u = u(t, x)$ be a bounded smooth solution to the Cauchy problem
\[
    u_t = u_{xx} \quad \text{for} \quad t > 0, \quad u(0, x) \equiv \exp(-x^4).
\]
Show that $u(t, 0)$ is not an analytic function at the point $t = 0$. 