Spring 2017. Math 5615H, Sec. 1
Homework 1 (due on Wednesday, February 15).

50 points are divided between 5 problems, 10 point each.

#1 (Ch.5, #15). Suppose that \( f \) is a twice-differentiable real function on \((a, \infty)\), and \(M_0, M_1, M_2\) are the least upper bounds of \(|f(x)|, |f'(x)|, |f''(x)|\), respectively, on \((a, \infty)\). Prove that \( M_2^2 \leq 4M_0M_2 \).

#2 (Ch.5, #17). Suppose \( f \) is a real, three times differentiable function on \([-1, 1]\), such that
\[
\begin{align*}
f(-1) &= 0, \quad f(0) = 0, \quad f(1) = 1, \quad f'(0) = 0.
\end{align*}
\]
Prove that \( f^{(3)}(x) \geq 3 \) for some point \( x \in [-1, 1] \). Note that the above equalities hold for \( f_0(x) := \frac{1}{2}(x^3 + x^2) \).

#3 (Ch.5, #26). Suppose \( f \) is differentiable on \([a, b]\), \( f(a) = 0 \), and there is a real number \( A \) such that \( |f'(x)| \leq A|f(x)| \) on \([a, b]\). Prove that \( f(x) \equiv 0 \) on \([a, b]\).

Hint. Consider the function \( g(x) := e^{-2Ax}f^2(x) \).

#4. Let \( f(x) \) be a continuously differentiable function on \([0, +\infty)\), such that
\[
\begin{align*}
f'(x) &= \cos(x^2) \cdot f(x) \quad \text{for all} \quad x \geq 0, \quad \text{and} \quad f(0) = 1.
\end{align*}
\]
Show that
\[
e^{-x} \leq f(x) \leq e^x \quad \text{for all} \quad x \geq 0.
\]

#5. Using two-sided estimate
\[
\frac{1}{n+1} < \ln \left(1 + \frac{1}{n}\right) < \frac{1}{n},
\]
show that the sequence
\[
s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n, \quad \text{where} \quad \ln x := \log_e x,
\]
is convergent.

Remark. The constant \( \gamma = \lim s_n = 0.577215665\ldots \) is called Euler’s constant.