Short Solutions to Homework #1.

#1. (12 points.) Reformulate the pig market problem (Example 1.1 on p.4 in the textbook), assuming that a pig gains $\alpha$ pounds per day, where $0 \leq \alpha \leq 10$.

(a). Find the best time $x$ to sell the pig and the maximum profit $P$, as functions of $\alpha$.

(b). Compute the sensitivities $S(x, \alpha)$ and $S(P, \alpha)$ at the point $\alpha = 5$.

(c). Now suppose that the cost to keep is proportional to the weight $w$ of the pig, with the initial cost of $0.45$ a day, i.e. it is $0.45w/200$ dollars a day. Under this assumption, find the best time $x$ to sell the pig and the maximum profit $P$ for $\alpha = 5$.

Solution: #1 (a). The profit as a function of time $t$ (in days) is

$$f(t) = (0.65 - 0.01 \cdot t) \cdot (200 + \alpha t) - 0.45 \cdot t = 0.65 \alpha t - 0.01 \alpha t^2 - 2.45 t + 130.$$ 

From the equality $f'(x) = 0.65 \alpha - 0.02 \alpha x - 2.45 = 0$ we get

$$x = \frac{50.0}{\alpha} \cdot (0.65 \alpha - 2.45) = \frac{32.5 - 122.5}{\alpha} > 0 \quad \text{for} \quad \alpha > \frac{122.5}{32.5} \approx 3.769,$n

and $x = 0$ for $\alpha \leq 3.769$. More carefully, we have to consider the neighbor integers near $x$. Correspondingly, the maximal profit is

$$P = f_{\text{max}} = f(x) = 10.5625 \alpha + 50.375 + \frac{150.0625}{\alpha}$$ 

for $\alpha \geq 3.769$, and $P = f_{\text{max}} = f(0) = 130$ for $\alpha < 3.769$. In particular, for $\alpha = 5$, we get

$$x = 32.5 - \frac{122.5}{5} = 8, \quad P = 10.5625 \cdot 5 + 50.375 + \frac{150.0625}{5} = 133.2.$$ 

#1 (b). The sensitivities at the point $\alpha = 5$,

$$S(x, \alpha) = \frac{dP}{d\alpha} \cdot \frac{\alpha}{x} = \frac{122.5}{\alpha^2} \cdot \frac{\alpha^2}{32.5 \alpha - 122.5} = \frac{122.5}{32.5 \alpha - 122.5} = \frac{122.5}{32.5 \cdot 5 - 122.5} = 3.0625.$$

$$S(P, \alpha) = \frac{dP}{d\alpha} \cdot \frac{\alpha}{P} = \left(10.5625 - \frac{150.0625}{\alpha^2}\right) \cdot \frac{\alpha^2}{10.5625 \alpha^2 + 50.375 \alpha + 150.0625} = \frac{169 \alpha^2 - 2401}{169 \alpha^2 + 806 \alpha + 2401} = \frac{169 \cdot 5^2 - 2401}{169 \cdot 5 + 806 \cdot 5 + 2401} = 0.171171.$$ 

#1 (c). In this part, the revenue for selling the pig after $t$ days is the same as in the original formulation:

$$R = R(t) = (0.65 - 0.01 \cdot t) \cdot w(t), \quad \text{where} \quad w(t) = 200 + 5t \quad \text{- the weight of the pig}.$$ 

However, the cost of keeping the pig for $t$ days is now different:

$$C = C(t) = \sum_{s=1}^{t} \frac{0.45 \cdot w(s)}{200} = 0.45 \sum_{s=1}^{t} \left(1 + \frac{s}{40}\right) = 0.45 \left(t + \frac{t(t + 1)}{80}\right) = 0.005625 \cdot t^2 + 0.455625 \cdot t.$$ 

Hence the profit for selling the pig after $t$ days is

$$f(t) = R(t) - C(t) = (0.65 - 0.01 \cdot t) \cdot (200 + 5t) - \left(0.005625 \cdot t^2 + 0.455625 \cdot t\right) = -0.055625 \cdot t^2 + 0.794375 \cdot t + 130.0.$$ 

The best time to sell the pig is the minimal integer $x \geq 0$ for which $f(x + 1) - f(x) < 0$. We have

$$f(x + 1) - f(x) = -0.055625(2x + 1) + 0.794375 = 0.73875 - 0.11125x < 0 \quad \text{for} \quad x > 6.64045.$$
Therefore, the optimal time to sell the pig $x = 7$, and the corresponding profit

$$P = f(7) = -0.055 \cdot 7^2 + 0.794 \cdot 7 + 130.0 \approx 132.835.$$ 

In comparison with the original formulation, when $x = 8$ and the maximal profit was 133.2, we lose only 0.365, i.e. 36.5 cents. This is not surprising, because we lose 0.05 for selling the pig one day earlier, plus costs for "additional weight" = $0.45 \cdot (5 + 10 + 15 + 20 + 25 + 30 + 35)/200 = 0.315$.

#2. (8 points.) Solve Problem 1.4.9 (c) on p.18 in the textbook for general $n$, and apply the results to parts (a) and (b) in the case $n = 5,000$.

**Solution.** Assuming linearity, for the price $p \geq 1.5$ (in dollars), the number of subscribers is

$$N(p) = 80,000 - 10n \cdot (p - 1.5) = 80,000 - 10np + 15n,$$

and the profit

$$P = f(p) = N(p) \cdot p = 80,000p - 10np^2 + 15np.$$ 

Since

$$\frac{df(x)}{dx} = 80,000 - 20nx + 15n = 0 \quad \text{for} \quad x = \frac{4000}{n} + 0.75,$$

the optimal price

$$p = p(n) = \frac{4000}{n} + 0.75.$$ 

The sensitivity

$$S(p, n) = \frac{dp}{dn} \cdot \frac{n}{p} = -\frac{4000}{n^2} \cdot n \cdot \left(\frac{4000}{n} + 0.75\right)^{-1} = -\frac{16,000}{3n + 16,000}.$$ 

In the case $n = 5000$, we get

$$p = 0.8 + 0.75 = 1.55, \quad S(p, n) = S(p, 5000) = -\frac{16}{31} \approx -0.51613,$$

Profit $P = f(1.55) = 80,000 \cdot 1.55 - 10 \cdot 5,000 \cdot 1.55^2 + 15 \cdot 5,000 \cdot 1.55 = 12,0125$.

#3. (10 points.) In Example 2.1 on p.21 in the textbook, consider an imaginary situation with 0.3 and 0.4 cents being replaced by 3 and 4 cents correspondingly. This means that instead of (2.2)–(2.3) you now have to maximize

$$f(x_1, x_2) = (339 - 0.01x_1 - 0.03x_2)x_1 + (390 - 0.04x_1 - 0.01x_2)x_2 - (400,000 + 195x_1 + 225x_2)$$

over the set $S = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$.

**Solution.** We have

$$f(x_1, x_2) = (144 - 0.01x_1 - 0.03x_2)x_1 + (174 - 0.04x_1 - 0.01x_2)x_2 - 400,000.$$ 

The necessary condition for the interior extremum is

$$\frac{\partial f}{\partial x_1} = 144 - 0.07x_2 - 0.02x_1 = 0, \quad \frac{\partial f}{\partial x_2} = 174 - 0.02x_2 - 0.07x_1 = 0.$$ 

From this system we find $x_1 \approx 2,066.67, x_2 \approx 1,466.67$, and $f(x_1, x_2) = -123,600$.

However, in this case, the graph of $y = f(x_1, x_2)$ is a hyperbolic paraboloid, and the maximum is attained for $x_1 = 0$. Indeed, set $x = x_1 + x_2$. Then

$$f(x_1, x_2) \leq (174 - 0.01x_1 - 0.01x_2)x_1 + (174 - 0.01x_1 - 0.01x_2)x_2 - 400,000$$

$$= (174 - 0.01x)x - 400,000 = f(0, x).$$ 

The maximum of $f(0, x)$ is attained for $x = 8,700$ and $f_{max} = 356,900$.

#4. (10 points.) (a). Find the maximal possible volume $V$ of a cylinder

$$C = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 \leq r^2, 0 \leq x_3 \leq h\}$$

with the given total surface area $S$. 

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(b). Same question for a parallelepiped

\[ P = \{(x_1, x_2, x_3) : 0 \leq x_1 \leq a, x_2 \leq b, 0 \leq x_3 \leq c\} \]

instead of a cylinder \( C \).

**Solution. #4(a).** Volume \( V = f(r, h) = \pi r^2 h \), the surface area \( S = g(r, h) = 2\pi (r + h) \). We have

\[
\nabla f = \lambda \nabla g \implies 2\pi rh = \lambda (4\pi r + 2\pi h), \quad \pi r^2 = \lambda \cdot 2\pi r \implies r = 2\lambda, h = 2\lambda;
\]

\[ S = 6\pi r^2, \quad V_{\text{max}} = 2\pi r^3 = 2\pi (S/6)^{3/2} = \frac{S^{3/2}}{3\sqrt{3\pi}}. \]

**#4(b).** Volume \( V = f(a, b, c) = abc \), the surface area \( S = g(a, b, c) = 2(ab + bc + ac) \). We have

\[
\nabla f = \lambda \nabla g \implies bc = 2\lambda(b + c), \quad ac = 2\lambda(a + c), \quad ab = 2\lambda(a + b),
\]

\[ \implies 2\lambda = \frac{1}{b} + \frac{1}{c} = \frac{1}{a} + \frac{1}{c} = \frac{1}{b} + \frac{1}{a} \implies \frac{1}{a} = \frac{1}{b} = \frac{1}{c}.
\]

\[ \implies a = b = c = \left(\frac{S}{6}\right)^{1/2} \implies V_{\text{max}} = \left(\frac{S}{6}\right)^{3/2}. \]

**#5.** (10 points.) Solve Problem 2.4.6 (a) and (b) on p.52 in the textbook.

**Solution. (a).** We have two unknowns:

0 \leq x - \text{the price reduction, and } 0 \leq y \leq 50,000 - \text{additional advertising budget.}

By linearity assumptions, the sale increase because of the price reduction is \( c_1 x \), where the constant \( c_1 \) is determined from the equality \( c_1 \cdot 100 = 0.5 \), i.e. \( c_1 = 0.005 \).

Similarly, the sale increase because of additional advertising is \( c_2 y \), where \( c_2 \cdot 10,000 = 200 \), i.e. \( c_2 = 0.02 \). Now

The revenue \( R = R(x, y) = [(1 + 0.005) x] \cdot 10,000 + 0.02 y \). \( : (950 - x) \).

The cost \( C = C(x, y) = [(1 + 0.005) x] \cdot 10,000 + 0.02 y \). \( : 700 + 50,000 + y. \)

The profit \( P = f(x, y) = R(x, y) - C(x, y) = 2,500 x - 50 x^2 + 4 y - 0.02 xy + 2,450,000. \)

From the system \( \partial f/\partial x = 2,500 - 100 x - 0.02 y = 0, \quad \partial f/\partial y = 4 - 0.02 x = 0, \) we find: \( x = 200, \ y = -875,000.0 \). The point \( (x, y) \) does not satisfy the given restrictions. Therefore, we need to check the boundary points.

(i) \( x = 0. \) Then \( P = f(0, y) = 4y + 2,450,000 \) is maximal for \( y = 50,000 \), and it equals to \( P_1 = 2,650,000. \)

(ii) \( y = 0. \) Then \( P = f(x, 0) = 2,500 x - 50 x^2 + 2,450,000 = 50 x (50 - x) + 2,450,000 \) is maximal for \( x = 25 \), and it equals to \( P_2 = 2,481,250. \)

(iii) In the remaining case \( x > 0, \ g(y) = y = 50,000, \) we will use the Lagrange system \( \nabla f = \lambda \nabla g. \) In fact, we only need the first component of this vector equality: \( \partial f/\partial x = 2500 - 100 x - 0.02 y = \lambda \cdot \partial g/\partial x = 0. \) Since \( y = 50,000, \) we get \( x = 15, \) and \( P_{\text{max}} = f(x = 15, y = 50,000) = 2,661,250. \)

(b). If we replace \( 50\% = 0.5 \) by a parameter \( a, \) then we get \( c_1 = 0.01 a, \) which gives us

\[ P = f(x, y) = (25,000 a - 10,000)x - 100 ax^2 + 4y - 0.02 xy + 2,450,000. \]

We know that for \( a = 0.5, \) the maximum of \( f \) is attained on the boundary \( y = 50,000. \) By continuity, same holds true for \( a \) close to \( 0.0. \) Therefore, it suffices to consider \( f \) for \( y = 50,000. \) Then maximum is attained at the point

\[ x = 125 - \frac{55}{a}, \quad \text{and the sensitivity} \quad S(x, a) = \frac{dx}{da} \cdot x = \frac{55}{a^2} \cdot \left(\frac{a^2}{125a - 55} \right) = \frac{11}{25a - 11} = \frac{11}{25 \cdot 0.5 - 11} = \frac{22}{3} = 7.333... \]

Finally, since \( y = \text{const} = 50,000, \) we also have \( S(y, a) = 0 \) for \( a \) close to \( 0.5. \)