Math 4428, Sec. 2. MATHEMATICAL MODELING. Spring 2017

Solutions to Homework #4.

#1. (15 points.) Solve Problem 7.5.6, parts (a) and (b) on p.243 of the textbook. The random variable \( N_t \) has a Poisson distribution with parameter \( \lambda t \). Specifically,

\[
P\{N_t = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad \text{for all} \quad n = 0, 1, 2, \ldots.
\]

(a) Show that \( E(N_t) = \lambda t \) and \( \text{Var}(N_t) = \lambda t \).

(b). Use the Poisson distribution to calculate the probability that the number of calls received in a given month deviates from the mean of 171 by as much as 18 calls.

Solution (a). Denote \( X = N_t \), \( \mu = \lambda t \). The moment generating function of \( N_t \),

\[
\varphi(s) = E(e^{sX}) = \sum_{n=0}^{\infty} \frac{e^{sn} \cdot e^{-\mu} \mu^n}{n!} = e^{-\mu} \sum_{n=0}^{\infty} \frac{(e^s\mu)^n}{n!} = e^{-\mu} \cdot \exp(e^s\mu) = \exp([e^s-1]\mu).
\]

Note that \( \varphi(0) = 1, \varphi'(0) = E(X), \varphi''(0) = E(X^2) \).

It is convenient to consider the function \( \psi(s) = \ln \varphi(s) = (e^s-1)\mu \), which satisfies

\[
\psi'(s) = \frac{\varphi'(s)}{\varphi(s)} = e^s \mu, \quad \psi''(s) = \frac{\varphi''(s)\varphi(s) - [\varphi'(s)]^2}{\varphi^2(s)} = e^s \mu.
\]

In particular,

\[
\psi'(0) = \varphi'(0) = E(X) = \mu,
\]

\[
\psi''(0) = \varphi''(0) - [\varphi'(0)]^2 = E(X^2) - [E(X)]^2 = \text{Var}(X) = \mu.
\]

(b). We have \( \lambda t = \mu = 171 \). The corresponding probability

\[
P\{|X - \mu| \geq 18\} = 1 - P\{|X - \mu| \leq 17\} = 1 - \sum_{k=154}^{188} \frac{171^k \cdot e^{-171}}{k!} \approx 0.1806 \ldots
\]

For the Poisson distribution we have \( \mu = \sigma^2 = 171 \). Since this is a large number, we can expect that \( Y = (X-\mu)/\sigma \) is close to the standard normal distribution \( N(0,1) \) with density \( f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \), in the sense that

\[
P\{|X-\mu| \leq 17\} = P\{|Y| \leq 17/\sigma\} \approx 2 \int_0^{17/\sigma} f_Y(y) \, dy.
\]

Here \( 17.5/\sigma = 17.5/\sqrt{171} \approx 1.33826 \), hence

\[
P\{|X-\mu| \leq 17\} \approx \frac{2}{\sqrt{2\pi}} \int_0^{1.33826} e^{-y^2/2} \, dy \approx 0.8192,
\]

and the complementary probability

\[
P\{|X-\mu| \geq 18\} = 1 - P\{|X-\mu| \leq 17\} \approx 1 - 0.8192 = 0.1808.
\]

#2. (15 points.) Solve Problem 7.5.14, parts (a), (b), and (c) on p.247 of the textbook. The random variable \( X \) has the geometric distribution

\[
P\{X = i\} = p(1-p)^{i-1} \quad \text{for} \quad i = 1, 2, 3, \ldots, \quad \text{where} \quad 0 < p < 1.
\]
(a). Show that $P \{X > i\} = (1 - p)^i$.
(b). Show that the conditional probability $P \{X > i + j \mid X > j\} = P \{X > i\}$.
(c). Compute $E(X) = 1/p$.

Solution (a). Denote $q = 1 - p$.

$$P \{X > i\} = \sum_{j=i+1}^{\infty} pq^{j-1} = p \cdot (q^i + q^{i+1} + \cdots)$$

$$= pq^i \cdot (1 + q + q^2 + \cdots) = pq^i \cdot \frac{1}{1-q} = q^i.$$ 

(b). Since the event $A = \{X > i + j\} \subseteq B = \{X > j\}$, we have

$$P \{A \mid B\} = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} = \frac{q^{i+j}}{q^j} = q^i = P \{X > i\}.$$ 

(c). We have

$$E(X) = \sum_{i=1}^{\infty} i \cdot P \{X = i\} = p \sum_{i=1}^{\infty} i \cdot q^{i-1} = p \cdot d \sum_{i=0}^{\infty} q^i$$

$$= p \cdot d \frac{1}{(1-q)^{-1}} = p(1-q)^{-2} = p^{-1}.$$ 

#3. (20 points.) Solve Problem 8.5.6, parts (b) and (c) on p.292 of the textbook by Meerschaert. We have a Markov chain with five states $X_n = 0, 1, 2, 3, 4$ and the transition matrix

$$P = \begin{pmatrix}
0.9 & 0.05 & 0.025 & 0.015 & 0.01 \\
0.3 & 0.7 & 0 & 0 & 0 \\
0 & 0.4 & 0.6 & 0 & 0 \\
0 & 0 & 0.6 & 0.4 & 0 \\
0 & 0 & 0 & 0.8 & 0.2 \\
\end{pmatrix}$$

(b). Find the steady state probability distribution for this model.

(c). How often are severe floods (corresponding to state 4) expected to occur?

Solution (b). The steady state probabilities can be represented as a row vector $\pi$ satisfying $\pi = \pi P$, or $\pi (P - I) = 0$. Since $\text{det}(P - I) = 0$, the columns of $P - I$ are linearly independent. We must have $\sum \pi(j) = 1$. Replacing the last column of $P - I$ by $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T$, we get

$$\pi \cdot \begin{pmatrix}
-0.1 & 0.05 & 0.025 & 0.015 & 1 \\
0.3 & -0.3 & 0 & 0 & 1 \\
0 & 0.4 & -0.4 & 0 & 1 \\
0 & 0 & 0.6 & -0.6 & 1 \\
0 & 0 & 0 & 0.8 & 1 \\
\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$\pi = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix}
-0.1 & 0.05 & 0.025 & 0.015 & 1 \\
0.3 & -0.3 & 0 & 0 & 1 \\
0 & 0.4 & -0.4 & 0 & 1 \\
0 & 0 & 0.6 & -0.6 & 1 \\
0 & 0 & 0 & 0.8 & 1 \\
\end{pmatrix}^{-1}$$

$$\approx \begin{pmatrix} 0.661157, & 0.220386, & 0.082645, & 0.027548, & 0.008264 \end{pmatrix}.$$ 

(c). The frequency of this event is 0.008264, i.e. less than 1%.
**BONUS PROBLEM.** This is a supplementary problem which is beyond the minimum requirements for this class. You are not required to solve it, but if you decide to do it, please enclose the solution together with solutions to HW #4, but do not staple it together, because it will be graded separately by the instructor (not grader). You may get additional credit up to 20 points, depending on your creativity. This is a certain improvement of the diodes quality control test suggested in Example 7.1, pp. 223–227.

The suggested test consists of 3 stages. As in the textbook, we assume that diodes may be faulty with probability 0.003 each independently.

**I.** Start with a group test for \( n = 15 \) diodes in the same way as it is suggested in Example 7.1. The cost of this test is \( n + 4 = 19 \) cents.

**II.** If test I fails, then there is at least one faulty diode out of 15 under consideration. Divide these diodes into 3 subgroups with 5 diodes each, and perform a similar test for each of these subgroups, so that there will be 3 test, and each of them costs \( 5 + 4 = 9 \) cents.

**III.** Finally, perform individual test for each of subgroups for which test II fails. We may have 1, 2, or 3 such subgroups, and the cost of individual testing of each of them is \( 5 \times 5 = 25 \) cents.

Find the average testing cost per diode for the combined test (initial group test, subgroup tests, and individual tests), and compare it with \( A = 1.48 \) obtained in Example 7.1.

**Solution.** We consider this problem in a more general setting: a test group of \( n = mk \) diodes is subdivided into \( m \) subgroups with \( k \) diodes each. The cost of each individual connection is \( c_1 = 1 \) cent, and the cost of each individual test is \( c_2 = 5 \) cents. Each of diodes may be faulty with probability \( \varepsilon = 0.003 \) independently.

**I.** The cost of initial group test \( C_1 = (n - 1)c_1 + c_2 \). The probability of success is \( p_1 = (1 - \varepsilon)^n \). If this test fails (with probability \( q_1 = 1 - p_1 \)), then we proceed to steps II and III.

**II.** In a similar way, test each of \( m \) subgroups. Let \( X \) denote the number of subgroups with faulty diodes. The probability of success for each of these subgroups is \( p_2 = (1 - \varepsilon)^k \), and the probability of failure is \( q_2 = 1 - p_2 \). Because of independence, \( X \) has binomial distribution with parameters \( m \) and \( q_2 \), hence \( E(X) = mq_2 \). The random variable \( X \) can attain values 0, 1, 2, \ldots, \( m \), but we proceed to step II only if \( X \neq 0 \). Therefore, the total cost of testing in step II is

\[
C_2 = m \cdot ((k - 1)c_1 + c_2) \cdot I_{(X \neq 0)}, \quad \text{where} \quad I_{(X \neq 0)} = 1 \quad \text{if} \quad X \neq 0, \quad \text{and} \quad I_{(X \neq 0)} = 0 \quad \text{otherwise}.
\]

**III.** The total cost of individual testing of \( X \) subgroups with faulty diodes is \( C_3 = c_2 \cdot X \). The total cost \( C = C_1 + C_2 + C_3 \) has expectation \( E(C) = (n - 1)c_1 + c_2 + m \cdot ((k - 1)c_1 + c_2) \cdot P(X \neq 0) + c_2 \cdot E(X) \).

Since \( P(X \neq 0) = 1 - (1 - \varepsilon)^n \), \( E(X) = mq_2 = m \cdot (1 - (1 - \varepsilon)^k) \), and \( n = mk \), the average cost per diode is

\[
A = \frac{E(C)}{n} = 1 + \frac{c_2 - c_1}{n} + \left(1 + \frac{c_2 - c_1}{k}\right) \cdot (1 - (1 - \varepsilon)^n) + c_2 \cdot (1 - (1 - \varepsilon)^k).
\]

In our case \( n = 15, m = 3, k = 5, c_1 = 1 \), and \( c_2 = 5 \), we have

\[
A(n = 15, k = 5) = \frac{19}{15} + \frac{9}{5} \cdot (1 - 0.997^{15}) + 5 \cdot (1 - 0.997^5) \approx 1.4205.
\]

We can go further:

\[
A(n = 25, k = 5) = \frac{29}{25} + \frac{9}{5} \cdot (1 - 0.997^{25}) + 5 \cdot (1 - 0.997^5) \approx 1.3648,
\]

\[
A(n = 30, k = 5) = \frac{34}{30} + \frac{9}{5} \cdot (1 - 0.997^{30}) + 5 \cdot (1 - 0.997^5) \approx 1.3630,
\]

etc.