Math 4428: Mathematical Modeling: Spring 2017

Appendix D. Generating functions

Let \( X \) be a random variable with values in \( \mathbb{R}^1 \). The \textit{generating}, or \textit{probability generating function} of \( X \) is defined as \( \phi(t) = \phi_X(t) = E(t^X) \). If \( X \) has discrete distribution with \( p_k = P(X = k) \), \( k = 0, 1, 2, \ldots \), then

\[
\phi(t) = \sum_{k=0}^{\infty} p_k t^k \quad \text{for} \quad |t| \leq 1, \quad \text{and} \quad p_k = \frac{\phi^{(k)}(0)}{k!}.
\]

In this case, we also have

\[
\phi(1) = 1, \quad \phi^{(k)}(1) = E(X(X - 1) \cdots (X - k + 1)) \quad \text{for} \quad k = 1, 2, \ldots.
\]

The \textit{moment generating function} of \( X \) is \( \phi(t) = \varphi_X(t) = E(e^{tX}) = \phi(e^t) \). If it is defined in a neighborhood of the point \( t = 0 \), then the \( k \)\textsuperscript{th} moment of \( X \),

\[
E(X^k) = \varphi^{(k)}(0), \quad k = 1, 2, \ldots.
\]

In this case, we also have

\[
E(X) = \psi'(0), \quad \text{Var}(X) = \psi''(0), \quad \text{where} \quad \psi(t) = \ln \varphi(t).
\]

Note that if \( X_1, X_2, \ldots, X_n \) are independent, and \( X = X_1 + X_2 + \cdots + X_n \), then

\[
\varphi_X = \varphi_{X_1} \cdot \varphi_{X_2} \cdots \cdot \varphi_{X_n}, \quad \psi_X = \psi_{X_1} + \psi_{X_2} + \cdots + \psi_{X_n}.
\]

1. \( X = \text{Binomial} (n, p) \) with \( n = 1, 2, \ldots; 0 \leq p \leq 1 \).

\[
f_1(k) = P(X = k) = \binom{n}{k} p^k q^{n-k} \quad \text{for} \quad k = 0, 1, \ldots, n; \quad \text{where} \quad q = 1 - p;
\]

\[
\phi_1(t) = E(t^X) = (pt + q)^n, \quad \varphi_1(t) = E(e^{tX}) = (p e^t + q)^n, \quad \mu_1 = E(X) = np, \quad \sigma_1^2 = \text{Var}(X) = npq.
\]

2. \( X = \text{Poisson} (\lambda) \) with \( \lambda > 0 \).

\[
f_2(k) = P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad \text{for} \quad k = 0, 1, 2 \ldots;
\]

\[
\phi_2(t) = E(t^X) = \exp \left( \lambda(t - 1) \right), \quad \varphi_2(t) = E(e^{tX}) = \exp \left( \lambda(e^t - 1) \right), \quad \mu_2 = E(X) = \lambda, \quad \sigma_2^2 = \text{Var}(X) = \lambda.
\]

3. \( X = \text{Negative Binomial} (r, p) \) - the number of trials with probability of success \( p \) until \( r \)\textsuperscript{th} success.

\( Y = X - r = \text{Shifted Negative Binomial} (r, p) \) - the number of failures before \( r \)\textsuperscript{th} success. We have

\[
P(X = k) = \binom{k-1}{r-1} p^r q^{k-r} \quad \text{for} \quad k = r, r + 1, \ldots;
\]

\[
P(Y = j) = P(X = r + j) = \binom{r+j-1}{j} p^r q^j = \binom{r+j-1}{j} p^r q^j \quad \text{for} \quad j = 0, 1, 2, \ldots.
\]

Note that by Taylor’s formula, for \( a \in \mathbb{R}^1 \) and \( |t| < 1 \),

\[
g(t) = (1 + t)^a = 1 + \sum_{k=1}^{\infty} \frac{g^{(k)}(0)}{k!} t^k = 1 + \sum_{k=1}^{\infty} \frac{a}{k} t^k, \quad \text{where} \quad \frac{a}{k} = \frac{a(a-1) \cdots (a - k + 1)}{k!}.
\]
If \( a = n \) is a natural number, then this equality is reduced to the binomial formula (for all \( t \in \mathbb{R}^1 \)):

\[
g(t) = (1 + t)^n = 1 + \sum_{k=1}^{n} \binom{n}{k} t^k, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.
\]

Substituting \( t = -q \) and \( a = -r \), we get

\[
1 = p^r (1 - q)^{-r} = p^r \cdot \left(1 + \frac{r}{1!} q + \frac{r(r+1)}{2!} q^2 + \cdots \right) = \sum_{j=0}^{\infty} P(Y = j).
\]

Note that the above distributions are well defined for all \( r \in \mathbb{R}^1 \). correspondingly,

\[
\phi(t) = E(t^Y) = \sum_{j=0}^{\infty} t^j P(Y = j) = \sum_{j=0}^{\infty} \left(\frac{r}{r} \right)^j \left(\frac{p}{1-q} \right)^r;
\]

\[
\varphi_Y(t) = E(e^{tY}) = \left(\frac{p}{1-q} \right)^r, \quad \varphi_X(t) = E(e^{tX}) = e^{tr} E(e^{tY}) = \left(\frac{p}{e^{tr} - q} \right)^r;
\]

\[
\mu_Y = E(Y) = \frac{rq}{p}, \quad \mu_X = E(X) = \mu_Y + r = \frac{r}{p}, \quad \sigma_X^2 = Var(X) = Var(Y) = \frac{rq}{p^2}.
\]

3a. Geometric \((p)\) = Negative Binomial \((1, p)\).

4. \( X = \text{Gamma} \left(\alpha, \lambda \right) \), where \( \alpha > 0, \lambda > 0 \), if it has density

\[
f_4(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\lambda x) \quad \text{for} \quad x > 0, \quad \text{and} \quad f_4(x) = 0 \quad \text{otherwise}.
\]

Here \( \Gamma(\alpha) \) denotes the Gamma function:

\[
\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp(-x) \, dx,
\]

which satisfies the properties

\[
\Gamma(\alpha+1) = \alpha \Gamma(\alpha), \quad \Gamma(n+1) = n!, \quad \Gamma(1/2) = \sqrt{\pi}.
\]

We have

\[
\varphi_4(t) = E(e^{tX}) = \left(\frac{\lambda}{\lambda-t} \right)^\alpha, \quad \mu_4 = E(X) = \frac{\alpha}{\lambda}, \quad \sigma_4^2 = \text{Var}(X) = \frac{\alpha}{\lambda^2}.
\]

4a. Exponential \((\lambda)\) = Gamma \((1, \lambda)\).

5. \( X = \text{Normal} \left(\mu, \sigma^2 \right) \) is related to \( Y = \text{Standard Normal} = \text{Normal} \left(0, 1\right) \) by the formula

\[
X = \mu + \sigma \cdot Y.
\]

The corresponding densities

\[
f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right), \quad f_X(x) = \frac{1}{\sigma} f_Y\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right].
\]

We have

\[
\varphi_Y(t) = E(e^{tY}) = e^{t^2/2}, \quad \varphi_X(t) = E(e^{tX}) = \exp\left(t \mu + \frac{t^2 \sigma^2}{2}\right);
\]

\[
\mu_Y = E(Y) = 0, \quad \sigma_Y^2 = \text{Var}(Y) = 1; \quad \mu_X = E(X) = \mu, \quad \sigma_X^2 = \text{Var}(X) = \sigma^2.
\]