Appendix A. Simplex Method

Consider the problem of maximization of the objective function
\[ y = f(x) = c^T x = (c, x) = c_1x_1 + \cdots + c_nx_n \]
on a feasible region \( S \) defined by the constraints
\[ Ax \leq b, \text{ i.e. } g_i(x) = a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i \quad (i = 1, \ldots, m), \text{ and } x_i \geq 0 \text{ for all } i. \]  \( (S) \)

Introducing the auxiliary (slack) variables \( x_{n+1} = b_i - g_i(x) \geq 0 \), we can write
\[ g_i(x) = a_{i1}x_1 + \cdots + a_{in}x_n + x_{n+1} = b_i \quad (i = 1, \ldots, m). \]
The auxiliary variables also help to find an initial feasible solution in the form
\[ x^{(0)} = (0, \ldots, 0, b_1, \ldots, b_m)^T \in R^{m+n}. \]

In the tabular form, this problem corresponds to the table
\[
T = \begin{bmatrix}
1 & -c^T & 0 \\
0 & A & b
\end{bmatrix},
\]
where the original vector \( c \in R^n \) is extended to \( c = (c_1, \ldots, c_n, 0, \ldots, 0) \in R^{m+n} \), and the original \( m \times n \) matrix \( A \) is extended by the unit \( m \times m \) matrix \( I_m \), so that the resulting matrix \( A \) has size of \( m \times (m + n) \). It is convenient to numerate rows and columns of \( T \) starting from 0, then \( 0^{th} \) column corresponds to \( y \), and \( 0^{th} \) row – to the objective function.

We call \( x^{(0)} = (x_1^{(0)}, \ldots, x_{m+n}^{(0)})^T \in S \subset R^{m+n} \) a corner point of \( S \) if there is an ordered subset \( J = \{j_1, \ldots, j_m\} \) of distinct elements of \( \{1, 2, \ldots, m + n\} \) (not necessarily in increasing order), such that the corresponding columns \( A_j, j \in J \), of \( A \), are linearly independent, and \( x_j^{(0)} = 0 \) for all \( j \notin J \). The variables \( x_j, j \in J \) are called basic because \( A_j, j \in J \) compose a basis in \( R^m \). In addition if \( x_j^{(0)} > 0 \) for all \( j \in J \), then \( x^{(0)} \) is called a non-degenerate corner point of \( S \). Note that a weaker assumption \( x_j^{(0)} \geq 0 \) for all \( j \) is automatically true for \( x^{(0)} \in S \). Applying the elementary row operations (ERO) by the Gauss-Jordan procedure, or equivalently, calculating
\[
\begin{bmatrix}
1 & -c_{j_1} & \cdots & -c_{j_m} \\
0 & A_{j_1} & \cdots & A_{j_m}
\end{bmatrix}^{-1} T,
\]
we reduce the problem to the canonical form, when
\[ c_j = 0 \quad \text{for all } j \in J, \text{ and } \begin{bmatrix} A_{j_1} & \cdots & A_{j_m} \end{bmatrix} = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} = I_m - \text{ the unit } m \times m \text{ matrix}. \]  \( (C) \)

Since \( x_j^{(0)} = 0 \) for all \( j \notin J \), we have \( \sum_j c_j x_j^{(0)} = 0 \), hence 0 in the right column of \( T \) will be replaced by the value \( y^{(0)} \) of the objective function which we have to maximize, and \( b \) – by
\[ \sum_j x_j^{(0)} A_j = \sum_{k=1}^m x_k^{(0)} e_k = (x_1^{(0)}, \ldots, x_m^{(0)})^T. \]

Note that all component of resulting vector \( b \) are strictly positive if \( x^{(0)} \) is a non-degenerate corner point. Now consider the possible cases.

(1) \( c_k \leq 0 \) for all \( k \notin J \). Then for all \( x \in S \) we have \( y \leq y - \sum_j c_j x_j = \sum_j c_j^{(0)} x_j^{(0)} = y^{(0)} \), and \( x^* = x^{(0)} \) is the optimal solution.
(II) There exists \( k \notin J \) such that \( c_k > 0 \) and the column \( A_k \leq 0 \). Then considering \( x = x^{(0)} + t e_k \) with large \( t > 0 \), we see that the value of \( y \) can be made arbitrarily large, so there is no optimal solution.

(III) In this remaining case, there exists \( k \notin J \) such that \( c_k > 0 \), but \( A_k \leq 0 \) fails, i.e. some of \( a_{ik} > 0 \). Such a choice of \( k \) may be not unique. Then assuming that \( x^{(0)} \) is non-degenerate, choose the pivot element \( a_{rk} \) with \( r \) taking according to

\[
\min_{(i: \ a_{ik} > 0)} \frac{b_i}{a_{ik}} = \frac{b_r}{a_{rk}} > 0. \quad \text{(P)}
\]

Using this pivot element for the ERO, we make \( x_k \) the new basic (entering) variable in place of \( x_{j_r} \). Correspondingly, \( j_r \in J \) is replaced by \( k \). Repeating this procedure, we eventually arrive to one of the previous cases (I) or (II), provided all the intermediate corner point are non-degenerate.

Example 3.4 in the textbook. We can take \( x^{(0)} = [0, 0, 0, 1000, 300, 625]^T \) as the initial corner point for

\[
T = \begin{bmatrix}
1 & -400 & -200 & -250 & 0 & 0 & 0 \\
0 & 3 & 1 & 1.5 & 1 & 0 & 0 \\
0 & 0.8 & 2 & 0.3 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

Choose \( k = 1 \). According to (P), from the relation

\[
\min \{1000/3, 300/0.8, 625\} = 1000/3,
\]
we get \( r = 1 \), so that the pivot element is \( a_{11} = 3 \). Applying the ERO, we obtain

\[
T^{(1)} = \begin{bmatrix}
1 & -400 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0.8 & 1 & 0 \\
0 & 1 & 0 & 1 \\
\end{bmatrix}^{-1} \cdot T = \begin{bmatrix}
1 & 0 & -200 & -50 & 1000 & 0 & 0 & 0 & 0 \\
0 & 1 & \frac{1}{3} & \frac{1}{2} & 1 & 0 & 0 & 1000 & 0 \\
0 & 0 & \frac{1}{2} & -\frac{1}{3} & -\frac{1}{3} & 1 & 0 & 0 & 100 \\
0 & 0 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 1 & 0 & \frac{875}{3} \\
\end{bmatrix},
\]

with the new corner point \( x^{(1)} = [1000/3, 0, 0, 0, 1000/3, 875/3]^T \). Now we can take \( k = 2 \). Since

\[
\min \{1000, 875/2\} = 875/2,
\]
we get \( r = 3 \), with the pivot element \( a_{32} = 2/3 \). The next iteration is

\[
T^{(2)} = \begin{bmatrix}
1 & 0 & -200 & 0 \\
0 & 1 & \frac{1}{3} & 0 \\
0 & 0 & \frac{1}{3} & 1 \\
0 & 0 & \frac{2}{3} & 0 \\
\end{bmatrix}^{-1} \cdot T^{(1)} = \begin{bmatrix}
1 & 0 & 0 & 0 & 100 & 0 & 100 & 162500 \\
0 & 1 & 0 & 1 & \frac{1}{3} & 1 & 0 & \frac{375}{2} \\
0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 2 & \frac{875}{2} \\
0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 1 & 0 & \frac{125}{2} \\
\end{bmatrix},
\]

with the corner point \( x^{(2)} = [187.5, 437.5, 0, 0, 62.5, 0]^T \). Since in the last table we have \(-c^T \geq 0\), this is the case (I), which guarantees that

\[
x_1 = 187.5, \quad x_2 = 437.5, \quad x_3 = 0
\]

is the optimal solution with the maximal profit of $162,500, with the slack reserve \( x_5 = 62.5 \). These data match with results on p.79 in the textbook.