Final Exam is scheduled on Tuesday, May 12, 10:30 am – 13:30 pm.
For full credit, you need to show step-by-step calculations. Some reference material will be provided together with the assignment. There will be 6 problems (with possible subproblems) in the following 3 areas.

I. One variable and metavariable optimization. Lagrange multipliers (Ch. 1; Ch. 2, Sec. 2.1–2.2).

II. Differential Equations and Dynamical Systems. Stability of equilibrium points (Ch. 4, Sec. 4.1–4.2; Ch. 5, Sec. 5.1, 5.3).

III. Probability models (Ch. 7, Sec. 7.1, 7.2; Ch. 8, Sec. 8.1).

TYPICAL PROBLEMS with SOLUTIONS.

#I.1. Find a point $P(x,y)$ in the triangle

$$T = \{(x,y): x \geq 0, y \geq 0, x + y \leq 1\}$$

with vertices $O(0,0)$, $A(1,0)$, and $B(0,1)$, for which

$$f(x,y) = |OP|^2 + |AP|^2 + |BP|^2$$

is minimal.

**Solution.** We have $|OP|^2 = x^2 + y^2$, $|AP|^2 = (x - 1)^2 + y^2$, $|BP|^2 = x^2 + (y - 1)^2$, hence

$$f(x,y) = x^2 + y^2 + (x - 1)^2 + y^2 + x^2 + (y - 1)^2 = 3x^2 + 3y^2 - 2x - 2y + 2.$$ 

The absolute minimum of $f$ in the whole plane $Oxy$ is attained at the point $(x,y)$ satisfying the system

$$\frac{\partial f}{\partial x} = 6x - 2 = 0, \quad \frac{\partial f}{\partial y} = 6y - 2 = 0,$$

i.e. at the point $P_0(1/3, 1/3)$. Since this point lies in $T$ (it is the point of intersection of medians in $T$), the minimum of $f$ in $T$ is attained at $P_0(1/3, 1/3)$, and $f(P_0) = 4/3$.

Alternatively, one can also get this answer from the equality

$$f(x,y) = 3 \left(x - 1/3\right)^2 + 3 \left(y - 1/3\right)^2 + 4/3.$$ 

#I.2. Find the points on the graph of $y = x^2$ that are closest to the point $(0,3/2)$.

**Solution.** We need to find the minimum of the function $f(x,y) = x^2 + (y - 3/2)^2$, which is the square of distance between $(x,y)$ and $(0,3/2)$, over the set

$$S = \{(x,y): g(x,y) = x^2 - y = 0\}.$$ 

The Lagrange multiplier equation $\nabla f = \lambda \nabla g$ can be written in coordinates as the system $\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}, \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$, i.e.

$$2x = 2\lambda x, \; 2y - 3 = -\lambda,$$

in addition to $y = x^2$. 

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From the first equation it follows either (i) $x = 0$, or (ii) $\lambda = 1$.

(i) $x = 0 \implies y = 0$ with $f(0, 0) = 9/4$.

(ii) $\lambda = 1 \implies y = 1$ with $x = \pm 1$, $f(\pm 1, 1) = 5/4$.

Comparing the values of $f$ in the cases (i) and (ii), we see that $f$ is minimal at the points $(\pm 1, 1)$, and the point $(0, 0)$ is the point of local maximum of $f$ on the graph of $y = x^2$.

Alternatively, on the curve $y = x^2 \geq 0$, one can write

$$f(x,y) = x^2 + (y - 3/2)^2 = y^2 - 2y + 9/4 = (y - 1)^2 + 5/4,$$

with the same conclusion.

#II.1 Find the equilibrium points of the system

$$\frac{dx_1}{dt} = (x_1 - 1)(x_2 - 1),$$

$$\frac{dx_2}{dt} = x_1x_2 - 1,$$

and discuss their stability.

**Solution.** The equilibrium points are solutions of the system

$$F_1(x_1, x_2) = (x_1 - 1)(x_2 - 1) = 0, \quad F_2(x_1, x_2) = x_1x_2 - 1 = 0,$$

which has the only solution $P(1, 1)$. At this point, the matrix

$$A = \begin{pmatrix}
\frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\
\frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2}
\end{pmatrix} = \begin{pmatrix}
x_2 - 1 & x_1 - 1 \\
x_2 & x_1
\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}. $$

This matrix has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1$. Since one of eigenvalues is strictly positive, the equilibrium point $P(1, 1)$ is unstable.

#II.2. See Examples 1.1, 2.1, 2.2, and 3.1 in Appendix B: Ordinary Differential Equations.

#III.1. Let $X_1$ and $X_2$ be independent identically distributed (i.i.d) random variables with density

$$f_1(x) = f_2(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Find the densities of

$$Y_1 = \frac{1}{\sqrt{2}} (X_1 + X_2), \quad Y_2 = X_1^2 + X_2^2.$$

**Solution.** We can use the fact that the moment generating function (m.g.f.) of $X_1$ and $X_2$ is

$$\varphi_{X_1,2}(t) = E(e^{tX_1,2}) = e^{\frac{t^2}{2}}.$$

Then the m.g.f. of $Y_1$ is

$$E \left( e^{Y_1} \right) = E \left( e^{\frac{t}{\sqrt{2}} X_1} \cdot e^{\frac{t}{\sqrt{2}} X_2} \right) = E \left( e^{\frac{t^2}{2} X_1} \right) \cdot E \left( e^{\frac{t^2}{2} X_2} \right) = e^{\frac{t^2}{2}} \cdot e^{\frac{t^2}{2}} = e^{t^2}.$$
We have got the same m.g.f. as that of $X_1$ and $X_2$. By uniqueness, $Y_1$ must have same density as $X_1$ and $X_2$.

Further, using the fact that for independent $X_1$ and $X_2$ their joint density is $f_1(x_1)f_2(x_2)$, we can evaluate the distribution function of $Y_2$ for $y > 0$ as follows (here we use the polar coordinates):

$$F_{Y_2}(y) = P(Y_2 = X_1^2 + X_2^2 < y) = \int \int_{\{x_1^2 + x_2^2 < y\}} f_1(x_1)f_2(x_2) \ dx_1dx_2$$

$$= \frac{1}{2\pi} \int \int_{\{x_1^2 + x_2^2 < y\}} e^{-\frac{x_1^2 + x_2^2}{2}} \ dx_1dx_2$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^{\sqrt{y}} e^{-\frac{r^2}{2}} r \ dr = 1 - e^{-\frac{y}{2}}.$$  

The density of $Y_2$, 

$$f_{Y_2}(y) = \frac{d}{dy} F_{Y_2}(y) = \frac{1}{2} e^{-\frac{y}{2}} \text{ for } y > 0, \quad \text{and } f_{Y_2}(y) \equiv 0 \text{ for } y \leq 0.$$ 

This is the exponential distribution with parameter $\lambda = 1/2$.

#III.2. Consider the Markov chain with the state transition matrix 

$$P = \begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix}, \quad \text{where } 0 < \alpha, \beta < 1.$$ 

Find the the state transition matrix in $n$ steps 

$$P^n = \begin{pmatrix} p_{11}^{(n)} & p_{12}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)} \end{pmatrix} \quad \text{and the limits } \lim_{n \to \infty} p_{ij}^{(n)}.$$ 

**Solution. 1st way** (it can also be considered as a review of linear systems of differential equations. The matrix function 

$$X(t) = e^{tP} = \sum_{n=0}^{\infty} \frac{t^n}{n!} P^n \quad \text{is the unique solution of the problem } \frac{d}{dt} X = PX, \quad X(0) = I.$$ 

The characteristic equation of $P$, 

$$p(\lambda) = |\lambda I - P| = \begin{vmatrix} \lambda - 1 + \alpha & -\alpha \\ -\beta & \lambda - 1 + \beta \end{vmatrix} = (\lambda - 1)^2 + (\lambda - 1)(\alpha + \beta)$$

has roots $\lambda_1 = 1$ and $\lambda_2 = 1 - \alpha - \beta$. The corresponding eigenvectors are $v_1 = (1 \ 1)^T$ and $v_2 = (\alpha \ -\beta)^T$. Hence the vector functions $u_1(t) = e^{\lambda_1 t}v_1$ and $u_2(t) = e^{\lambda_2 t}v_2$ satisfy $u' = Pu$. The matrix function $\Phi(t) = \begin{pmatrix} u_1(t) & u_2(t) \end{pmatrix}$ with columns $u_1(t)$ and $u_2(t)$ also satisfies 

$$\Phi' = \begin{pmatrix} u_1' & u_2' \end{pmatrix} = \begin{pmatrix} Pu_1 & Pu_2 \end{pmatrix} = P \begin{pmatrix} u_1 & u_2 \end{pmatrix} = P\Phi.$$ 

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Finally, $Y(t) = \Phi(t)\Phi^{-1}(0)$ satiates

$$Y'(t) = \Phi'(t)\Phi^{-1}(0) = P\Phi(t)\Phi^{-1}(0) = PY(t) \quad \text{and} \quad Y(0) = I.$$ 

By uniqueness, we must have $Y(t) = X(t) = e^{tP}$, i.e.

$$e^{tP} = \sum_{n=0}^{\infty} \frac{t^n}{n!}P^n = \Phi(t)\Phi^{-1}(0) = \left( e^{\lambda_1 t} \begin{pmatrix} 1 & 0 \\ 1 & -\beta \end{pmatrix} \right) \left( \begin{pmatrix} 1 & \alpha \\ 1 & -\beta \end{pmatrix} \right)^{-1} \left( \begin{pmatrix} 1 & \alpha \\ 1 & -\beta \end{pmatrix} \right)^{-1}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left( \lambda_1^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} \lambda_2^n \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} \right) \left( \begin{pmatrix} 1 & \alpha \\ 1 & -\beta \end{pmatrix} \right)^{-1}.$$ 

Comparing the coefficient of $\frac{t^n}{n!}$, and using the fact that $\lambda_1 = 1$ and $|\lambda_2| < 1$, we get

$$P^n = \left( \lambda_1^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} \lambda_2^n \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} \right) \left( \begin{pmatrix} 1 & \alpha \\ 1 & -\beta \end{pmatrix} \right)^{-1} \rightarrow \left( \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & \alpha \\ 1 & -\beta \end{pmatrix} \right)^{-1} = \frac{1}{\alpha + \beta} \begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix}.$$ 

This means that $p^{(n)}_{ij} \rightarrow \pi_j$ as $n \rightarrow \infty$, where

$$\pi = (\pi_1 \pi_2) = \frac{1}{\alpha + \beta} \begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix} \text{ satisfies } \pi = \pi P \quad \text{and} \quad \pi_1 + \pi_2 = 1.$$ 

**2nd way.** This is a standard way in Linear Algebra. Denote $V = \Phi(0) = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$. Then from

$$PV = \begin{pmatrix} Pv_1 & Pv_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 v_1 & \lambda_2 v_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = VA$$

it follows $P = VA V^{-1}$, and

$$P^n = VA^n V^{-1} = \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \left( \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \right) \left( \begin{pmatrix} 1 & \alpha \\ 1 & -\beta \end{pmatrix} \right)^{-1}$$

which gives the same representation as above.

**#III.3.** Find the steady state distribution for the state transition matrix

$$P = \begin{pmatrix} 0 & q & 0 & p \\ q & 0 & p & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}, \quad \text{where } 0 < p < 1, \quad q = 1 - p.$$ 

**Solution.** The steady state distribution $\pi = (\pi_1 \pi_2 \pi_3 \pi_4)$ must satisfy $\pi = \pi P$, which is equivalent to $\pi(P - I) = 0$, and $\sum \pi_k = 1$. The equality $\pi(P - I) = 0$ can be written as a system

$$-\pi_1 + q\pi_2 + \pi_3/2 + \pi_4/2 = 0,$$
$$q\pi_1 - \pi_2 + \pi_3/2 + \pi_4/2 = 0,$$
$$p\pi_2 - \pi_3 = 0,$$
$$p\pi_1 - \pi_4 = 0.$$
From the first two equalities it follows $\pi_1 = \pi_2 = a$, and from the last two $-\pi_3 = \pi_4 = pa$. In addition $\sum \pi_k = 1$ implies $2(1 + p)a = 1$, so that $a = [2(1 + p)]^{-1}$, and finally,

$$\pi = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix} = \frac{1}{2(1 + p)} \begin{pmatrix} 1 & 1 & p & p \end{pmatrix}.$$ 

**Alternatively**, one can notice that

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix} \begin{pmatrix} -1 & q & 0 & 1 \\ q & -1 & p & 1 \\ 1/2 & 1/2 & -1 & 1 \\ 1/2 & 1/2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix},$$

and get

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1-p & 0 & 1 \\ 1-p & -1 & p & 1 \\ 1/2 & 1/2 & -1 & 1 \\ 1/2 & 1/2 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2p+2} & \frac{1}{2p+2} & \frac{p}{2p+2} & \frac{p}{2p+2} \end{pmatrix},$$

i.e. we get the same answer.