Short Solutions to Homework #1.

#1. (12 points.) Reformulate the pig market problem (Example 1.1 on p.4 in the textbook), assuming that a pig gains $\alpha$ pounds per day, where $0 \leq \alpha \leq 10$.

(a). Find the best time $x$ to sell the pig and the maximum profit $P$, as functions of $\alpha$.

(b). Compute the sensitivities $S(x, \alpha)$ and $S(P, \alpha)$ at the point $\alpha = 5$.

(c). Now suppose that the cost to keep is proportional to the weight $w$ of the pig, with the initial cost of $0.45$ a day, i.e. it is $0.45w/200$ dollars a day. Under this assumption, find the best time $x$ to sell the pig and the maximum profit $P$ for $\alpha = 5$.

Remark. The parts (a) and (b) are basically contained on pages 11–13 in the textbook, with $g$ and $y$ in place of $\alpha$ and $P$, correspondingly.

Solution: #1 (a). The profit as a function of time $t$ (in days) is

$$f(t) = (0.65 - 0.01 \cdot t) \cdot (200 + \alpha t) - 0.45 \cdot t = -0.01 \alpha t^2 + 0.05 \cdot (13 \alpha - 49) t + 130.$$ 

From the equality $f'(x) = -0.02 \alpha x + 0.05 \cdot (13 \alpha - 49) = 0$ we get

$$x = \frac{5 \cdot (13 \alpha - 49)}{2 \alpha} = \frac{5}{2} \cdot \left(13 - \frac{49}{\alpha}\right) > 0 \quad \text{for} \quad \alpha > \alpha_0 := \frac{49}{13} \approx 3.769,$$

and $x = 0$ for $\alpha \leq \alpha_0$. More carefully, we have to consider the neighbor integers near $x$.

Then

$$f(t) = -0.01 \alpha \cdot (t - x)^2 + c, \quad \text{where} \quad c := 0.01 \alpha x^2 + 130.$$ 

Correspondingly, the maximal profit is

$$P = f_{\max} = f(x) = c = 0.01 \alpha x^2 + 130 = \frac{(13\alpha - 49)^2}{16\alpha} + 130 = \frac{169\alpha}{16} + \frac{403}{8} + \frac{2401}{16\alpha}.$$

and $P = f_{\max} = f(0) = 130$ for $\alpha < \alpha_0$. In particular, for $\alpha = 5$, we get $x = 8$ and $P = 133.2$.

#1 (b). The sensitivities at the point $\alpha = 5$,

$$S(x, \alpha) = \frac{dx}{d\alpha} \cdot \frac{\alpha}{x} = \frac{49}{13\alpha - 49} = \frac{49}{16} = 3.0625,$$

$$\frac{dP}{d\alpha} = \frac{1}{16} \cdot \left(169 - \frac{2401}{\alpha^2}\right) = \frac{1}{16} \cdot \left(169 - \frac{2401}{25}\right) = \frac{114}{25},$$

$$S(P, \alpha) = \frac{dP}{d\alpha} \cdot \frac{\alpha}{P} = \frac{114}{25} \cdot \frac{5}{133.2} = \frac{19}{111} \approx 0.171171.$$

#1 (c). In this part, the revenue for selling the pig after $t$ days is the same as in the original formulation:

$$R = R(t) = (0.65 - 0.01 \cdot t) \cdot w(t), \quad \text{where} \quad w(t) = 200 + 5t \quad \text{- the weight of the pig.}$$
However, the cost of keeping the pig for \( t \) days is now different:

\[
C = C(t) = \sum_{s=1}^{t} \frac{0.45 \cdot w(s)}{200} = 0.45 \sum_{s=1}^{t} \left(1 + \frac{s}{40}\right) = 0.45 \left( t + \frac{t(t+1)}{80} \right)
\]

\[
= 0.005625 \cdot t^2 + 0.455625 \cdot t.
\]

Hence the profit for selling the pig after \( t \) days is

\[
f(t) = R(t) - C(t) = (0.65 - 0.01 \cdot t) \cdot (200 + 5t) - \left(0.005625 t^2 + 0.455625 t\right)
\]

\[
= -0.055625 \cdot t^2 + 0.794375 \cdot t + 130.
\]

The best time to sell the pig is the minimal integer \( x \geq 0 \) for which \( f(x+1) - f(x) < 0 \). We have

\[
f(x+1) - f(x) = -0.055625 \cdot 2x + 0.794375 = 0.73875 - 0.11125x < 0 \quad \text{for} \quad x > 6.64045.
\]

Therefore, the optimal time to sell the pig \( x = 7 \), and the corresponding profit

\[
P = f(7) = -0.055625 \cdot 7^2 + 0.794375 \cdot 7 + 130.0 \approx 132.835.
\]

\#2. (8 points.) Solve Problem 1.4.9 (c) on p.18 in the textbook for general \( n \), and apply the results to parts (a) and (b) in the case \( n = 5,000 \).

**Solution.** Assuming linearity, for the price \( p \geq 1.5 \) (in dollars), the number of subscribers is

\[
N(p) = 80,000 - 10n \cdot (p - 1.5) = 80,000 - 10np + 15n,
\]

and the profit

\[
P = f(p) = N(p) \cdot p = 80,000 p - 10np^2 + 15np.
\]

Since

\[
\frac{df(x)}{dx} = 80,000 - 20nx + 15n = 0 \quad \text{for} \quad x = \frac{4000}{n} + 0.75,
\]

the optimal price

\[
p = p(n) = \frac{4000}{n} + 0.75.
\]

The sensitivity

\[
S(p,n) = \frac{dp}{dn} \cdot \frac{n}{p} = -\frac{4000}{n^2} \cdot n \cdot \left(\frac{4000}{n} + 0.75\right)^{-1} = -\frac{16,000}{3n + 16,000}.
\]

In the case \( n = 5000 \), we get

\[
p = 0.8 + 0.75 = 1.55, \quad S(p,n) = S(p,5000) = -\frac{16}{31} \approx -0.51613,
\]

Profit \( P = f(1.55) = 80,000 \cdot 1.55 - 10 \cdot 5,000 \cdot 1.55^2 + 15 \cdot 5,000 \cdot 1.55 \approx 12,0125.\)
#3. (10 points.) In Example 2.1 on p.21 in the textbook, consider an imaginary situation with 0.3 and 0.4 cents being replaced by 3 and 4 cents correspondingly. This means that instead of (2.2)–(2.3) you now have to maximize
\[ f(x_1, x_2) = (339 - 0.01x_1 - 0.03x_2)x_1 + (399 - 0.04x_1 - 0.01x_2)x_2 - (400,000 + 195x_1 + 225x_2) \]
over the set \( S = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\} \).

Solution. We have
\[ f(x_1, x_2) = (144 - 0.01x_1 - 0.03x_2)x_1 + (174 - 0.04x_1 - 0.01x_2)x_2 - 400,000. \]
The necessary condition for the interior extremum is
\[ \frac{\partial f}{\partial x_1} = 144 - 0.07x_2 - 0.02x_1 = 0, \quad \frac{\partial f}{\partial x_2} = 174 - 0.02x_2 - 0.07x_1 = 0. \]
From this system we find \( x_1 \approx 2,066.67, x_2 \approx 1,466.67 \), and \( f(x_1, x_2) = -123,600 \).

However, in this case, the graph of \( y = f(x_1, x_2) \) is a hyperbolic paraboloid, and the maximum is attained for \( x_1 = 0 \). Indeed, set \( x = x_1 + x_2 \). Then
\[ f(x_1, x_2) \leq (174 - 0.01x_1 - 0.01x_2)x_1 + (174 - 0.01x_1 - 0.01x_2)x_2 - 400,000 \]
\[ = (174 - 0.01x)x - 400,000 = f(0, x). \]
The maximum of \( f(0, x) \) is attained for \( x = 8,700 \) and \( f_{\text{max}} = 356,900 \).

#4. (10 points.) (a). Find the maximal possible volume \( V \) of a cylinder
\[ C = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 \leq r^2, 0 \leq x_3 \leq h\} \]
with the given total surface area \( S \).
(b). Same question for a parallelepiped
\[ P = \{(x_1, x_2, x_3) : 0 \leq x_1 \leq a, \leq x_2 \leq b, 0 \leq x_3 \leq c\} \]
instead of a cylinder \( C \).

Solution. #4(a). Volume \( V = f(r, h) = \pi r^2 h \), the surface area \( S = g(r, h) = 2\pi r (r + h) \). We have
\[
\nabla f = \lambda \nabla g \quad \implies \quad 2\pi rh = \lambda (4\pi r + 2\pi h), \quad \pi r^2 = \lambda \cdot 2\pi r \quad \implies \quad r = 2\lambda, \ h = 2r;
\]
\[ S = 6\pi r^2, \quad V_{\text{max}} = 2\pi r^3 = 2\pi (S/6\pi)^{3/2} = \frac{S^{3/2}}{3\sqrt{6\pi}}. \]

#4(b). Volume \( V = f(a, b, c) = abc \), the surface area \( S = g(a, b, c) = 2(ab + bc + ac) \). We have
\[
\nabla f = \lambda \nabla g \quad \implies \quad bc = 2\lambda (b + c), \ ac = 2\lambda (a + c), \ ab = 2\lambda (a + b),
\]
\[ \implies \quad 2\lambda = \frac{1}{b} + \frac{1}{c} = \frac{1}{a} + \frac{1}{c} = \frac{1}{b} + \frac{1}{a} \quad \implies \quad \frac{1}{a} = \frac{1}{b} = \frac{1}{c}
\]
\[ \implies \quad a = b = c = \left( \frac{S}{6} \right)^{1/2} \quad \implies \quad V_{\text{max}} = \left( \frac{S^{3/2}}{6} \right). \]
\#5. (10 points.) Solve Problem 2.4.6 (a) and (b) on p.52 in the textbook.

**Solution. (a).** We have two unknowns:

\[ 0 \leq x - \text{the price reduction, and } 0 \leq y \leq 50,000 - \text{additional advertising budget.} \]

By linearity assumptions, the sale increase because of the price reduction is \( c_1 x \), where the constant \( c_1 \) is determined from the equality \( c_1 \cdot 100 = 0.5 \), i.e. \( c_1 = 0.005 \).

Similarly, the sale increase because of additional advertising is \( c_2 y \), where \( c_2 \cdot 10,000 = 200 \), i.e. \( c_2 = 0.02 \). Now

The revenue \( R = R(x, y) = [(1 + 0.005 x) \cdot 10,000 + 0.02 y] \cdot (950 - x) \).

The cost \( C = C(x, y) = [(1 + 0.005 x) \cdot 10,000 + 0.02 y] \cdot 700 + 50,000 + y \).

The profit \( P = f(x, y) = R(x, y) - C(x, y) = 2,500 x - 50 x^2 + 4 y - 0.02 xy + 2,450,000 \).

From the system

\[ \partial f/\partial x = 2,500 - 100 x - 0.02 y = 0, \quad \partial f/\partial y = 4 - 0.02 x = 0, \]

we find: \( x = 200, \quad y = -875,000.0 \). The point \( (x, y) \) does not satisfy the given restrictions. Therefore, we need to check the boundary points.

(i) \( x = 0 \). Then \( P = f(0, y) = 4y + 2,450,000 \) is maximal for \( y = 50,000 \), and it equals to \( P_1 = 2,650,000 \).

(ii) \( y = 0 \). Then \( P = f(x, 0) = 2,500 x - 50 x^2 + 2,450,000 = 50 x (50 - x) + 2,450,000 \) is maximal for \( x = 25 \), and it equals to \( P_2 = 2,481,250 \).

(iii) In the remaining case \( x > 0 \), \( g(y) = y = 50,000 \), we will use the Lagrange system \( \nabla f = \lambda \nabla g \). In fact, we only need the first component of this vector equality:

\[ \partial f/\partial x = 2500 - 100 x - 0.02 y = \lambda \cdot \partial g/\partial x = 0. \]

Since \( y = 50,000 \), we get

\[ x = 15, \quad \text{and } P_{\text{max}} = f(x = 15, y = 50,000) = 2,661,250. \]

(b). If we replace \( 50\% = 0.5 \) by a parameter \( a \), then we get \( c_1 = 0.01a \), which gives us

\[ P = f(x, y) = (25,000 a - 10,000) x - 100 a x^2 + 4 y - 0.02 x y + 2,450,000. \]

We know that for \( a = 0.5 \), the maximum of \( f \) is attained on the boundary \( y = 50,000 \).

By continuity, same holds true for \( a \) close to 0.5. Therefore, it suffices to consider \( f \) for \( y = 50,000 \). Then maximum is attained at the point \( x = 125 - 55/a \), and the sensitivity

\[ S(x, a) = \frac{dx}{da} \cdot \frac{a}{x} = 55 \cdot \frac{a^2}{125 a - 55} = \frac{11}{25 a - 11} = \frac{11}{25 \cdot 0.5 - 11} = \frac{22}{3} = 7.333... \]

Finally, since \( y = \text{const} = 50,000 \), we also have \( S(y, a) = 0 \) for \( a \) close to 0.5.