Appendix B: Simplex Method.

Consider the problem of maximization of

\[ y = f(x) = c^T x = (c, x) = c_1x_1 + \cdots + c_nx_n \]

by the constraints

\[ Ax \leq b, \text{ i.e. } g_i(x) = a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i \quad (i = 1, \ldots, n), \text{ and } x_i \geq 0 \text{ for all } i. \]

Introducing the auxiliary (slack) variables \( x_{n+i} = b_i - g_i(x) \geq 0 \), we can write

\[ g_i(x) = a_{i1}x_1 + \cdots + a_{in}x_n + x_{n+i} = b_i \quad (i = 1, \ldots, n). \]

The auxiliary variables also help to find an initial feasible solution in the form

\[ x^{(0)} = (0, \ldots, 0, b_1, \ldots, b_n)^T \in \mathbb{R}^{m+n}. \]

Without loss of generality, we can assume that \( g_i(x) = b_i \geq 0 \) for all \( 1 \leq i \leq m \leq n \). In the tabular form, this problem corresponds to the table

\[
T = T := \begin{bmatrix}
1 & -c^T & 0 \\
0 & A & b
\end{bmatrix}
\]

We call \( x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)})^T \) a non-degenerate corner point if there are \( 1 \leq j_1 < j_2 < \cdots < j_m \leq n \) such that the corresponding columns \( A_{j_l} \) of \( A \) are linearly independent (i.e. \( x_{j_l} \) can be taken as the basic variables), \( x_{j_l}^{(0)} > 0 \) for all \( l \), and \( x_{k}^{(0)} = 0 \) for all \( k \neq j_l \). Applying the elementary row operations (ERO), or equivalently, calculating

\[
\begin{bmatrix}
1 & -c_{j_1} & \cdots & -c_{j_m} \\
0 & A_{j_1} & \cdots & A_{j_m}
\end{bmatrix}^{-1} \cdot T,
\]

we reduce the problem to the case

\[ c_{j_1} = \cdots = c_{j_m} = 0, \quad A_{j_1} \cdots A_{j_m} = I_m \text{- the unit } m \times m \text{ matrix}, \quad b = [x_{j_1}^{(0)}, \ldots, x_{j_m}^{(0)}]^T. \]

Now consider the possible cases:

(I) \( c_k \leq 0 \) for all \( k \neq j_l \). In this case, the corner point \( x^* = x^{(0)} \) is the optimal solution.

(II) \( \exists k \neq j_l \) such that \( c_k > 0 \), and \( a_{ik} \leq 0 \) for all \( i \). In this case, the function \( y = f(x) = c^T x \) is unbounded, it can have arbitrarily large values by taking large \( x_k > 0 \).

(III) \( \exists k \neq j_l \) such that \( c_k > 0 \), but some of \( a_{ik} > 0 \). In this case, choose \( r \) from the relation

\[
\min_{\{i: a_{ik} > 0\}} \frac{x_i^{(0)}}{a_{ik}} = \frac{x_r^{(0)}}{a_{rk}}.
\]

Using \( a_{rk} \) as a pivot element, make \( x_k \) the new basic variable instead of \( x_r \).
Example 3.4 in the textbook. We have

\[
T = \begin{bmatrix}
1 & -400 & -200 & -250 & 0 & 0 & 0 \\
0 & 0 & 3 & 1 & 1.5 & 1 & 0 & 0 & 1000 \\
0 & 0 & 0.8 & 0.2 & 0.3 & 1 & 0 & 300 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 625 \\
\end{bmatrix}
\]

\[
x^{(0)} = [0, 0, 0, 1000, 3000, 625]^T.
\]

Take \(k = 1\). From the relation

\[
\min \{1000/3, \, 3000/8, \, 625\} = 1000/3,
\]

we get \(r = 1\), so that the pivot element is \(a_{11} = 3\). Applying the ERO, we obtain

\[
T^{(1)} = \begin{bmatrix}
1 & 400/3 & 0 & 0 \\
0 & 1/3 & 0 & 0 \\
0 & -8/3 & 1 & 0 \\
0 & -1/3 & 0 & 1 \\
\end{bmatrix}
\cdot T = \begin{bmatrix}
1 & 0 & -200/3 & -50 & 400/3 & 0 & 0 \\
0 & 1 & 1/3 & 1/2 & 1/3 & 0 & 0 \\
0 & 0 & -1/15 & -1/10 & -4/15 & 1 & 0 \\
0 & 0 & 2/3 & 1/2 & -1/3 & 0 & 1 \\
\end{bmatrix}
\]

\[
x^{(1)} = [1000/3, 0, 0, 0, 1000/3, 875/3]^T.
\]

Now we can take \(k = 2\). Since

\[
\min \{1000, \, 875/2\} = 875/2,
\]

we get \(r = 3\), with the pivot element is \(a_{32} = 2/3\). The last iteration is

\[
T^{(2)} = \begin{bmatrix}
1 & 0 & -200/3 & 0 \\
0 & 1 & 1/3 & 0 \\
0 & 0 & -1/15 & 1 \\
0 & 0 & 2/3 & 0 \\
\end{bmatrix}^{-1}
\cdot T^{(1)} = \begin{bmatrix}
1 & 0 & 0 & 100 \\
0 & 1 & 0 & -1/2 \\
0 & 0 & 0 & 3/2 \\
0 & 0 & 1 & 1/10 \\
\end{bmatrix}
\cdot T^{(1)}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 & 0 & 100 & 0 & 100 & 162500 \\
0 & 1 & 0 & 1/4 & 1/2 & 0 & -1/2 & 375/2 \\
0 & 0 & 1 & 3/4 & -1/2 & 0 & 3/2 & 875/2 \\
0 & 0 & 0 & -1/20 & -3/10 & 1 & 1/10 & 125/2 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 1 & 0 & 0.25 & 0.5 & 0 & -0.5 & 187.5 \\
0 & 0 & 1 & 0.75 & -0.5 & 0 & 1.5 & 437.5 \\
0 & 0 & 0 & -0.05 & -0.03 & 1 & 0.1 & 62.5 \\
\end{bmatrix}
\]

guarantees that

\[
x_1 = 187.5, \quad x_2 = 437.5, \quad x_3 = 0
\]

is the optimal solution with the maximal profit of $162,500.