Spring 2019  
Math 4512: Differential Equations with Applications.

Solutions to Homework #5.

#1. Find the general solution of the equation

\[ Ly = y''' + y'' + 3y' - 5y = t^3 e^{-t} \sin t. \]

Solution. The characteristic equation

\[ p(r) = r^3 + r^2 + 3r - 5 = (r - 1) [(r + 1)^2 + 4] \]

has roots \( r_1 = 1, \quad r_{2,3} = -1 \pm 2i. \)

Hence the general solution is

\[ y(t) = c_1 e^t + e^{-t} (c_2 \cos 2t + c_3 \sin 2t) + y_p(t), \]

where \( y_p(t) \) is a particular solution. One can find \( y_p(t) \) in the form

\[ y_p = \text{Im} \ z_p, \quad \text{where} \quad L z_p = t^3 e^{\lambda t} \]

with \( \lambda := -1 + i. \)

Since \( \lambda \) is not a root of \( p(r) \), one can specify

\[ z_p(t) = e^{\lambda t} P_3(t), \quad \text{where} \quad P_3(t) := A_0 + A_1 t + A_2 t^2 + A_3 t^3 \]

with coefficients \( A_j. \)

In order to find the undetermined coefficients \( A_0, A_1, A_2, A_3, \) we will use the relation

\[ L z_p = p(D)(e^{\lambda t} P_3) = e^{\lambda t} p(D + \lambda) P_3 = e^{\lambda t} t^3, \quad \text{i.e.} \quad L_0 P_3 := p(D + \lambda) P_3 = t^3. \]

Here

\[ L_0 P_3 := p(D + \lambda) P_3 = \sum_{k=0}^{3} \frac{p^{(k)}(\lambda)}{k!} D^k P_3 =: (a_0 + a_1 D + a_2 D^2 + a_3 D^3) P_3; \]

\[
\begin{align*}
a_0 & := p(\lambda) = \lambda^3 + \lambda^2 + 3\lambda - 5 = -6 + 3i, \\
a_1 & := p'(\lambda) = 3\lambda^2 + 2\lambda + 3 = 1 - 4i, \\
a_2 & := \frac{1}{2} p''(\lambda) = 3\lambda + 1 = -2 + 3i, \\
a_3 & := \frac{1}{6} p'''(\lambda) = 1.
\end{align*}
\]

The standard way to find the coefficients \( A_j \) is to plug the expression for \( P_3(t) \) in (3) into the equality \( L_0 P_3 = t^3 \):

\[ t^3 = (a_0 A_0 + a_1 A_1 + 2a_2 A_2 + 6a_3 A_3) + (a_0 A_1 + 2a_1 A_2 + 6a_2 A_3) t + (a_0 A_2 + 3a_1 A_3) t^2 + a_0 A_3 t^3. \]

Comparing the coefficient, we get a system with triangular matrix, which can be written in the matrix form:

\[
\begin{pmatrix}
a_0 & a_1 & 2a_2 & 6a_3 \\
0 & a_0 & 2a_1 & 6a_2 \\
0 & 0 & a_0 & 3a_1 \\
0 & 0 & 0 & a_0 \\
\end{pmatrix}
\begin{pmatrix}
A_0 \\
A_1 \\
A_2 \\
A_3 \\
\end{pmatrix}
= 
\begin{pmatrix}
-6 + 3i & 1 - 4i & -4 + 6i & 6 \\
0 & -6 + 3i & 2 - 8i & -12 + 18i \\
0 & 0 & -6 + 3i & 3 - 12i \\
0 & 0 & 0 & -6 + 3i \\
\end{pmatrix}
\begin{pmatrix}
A_0 \\
A_1 \\
A_2 \\
A_3 \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
\end{pmatrix}.
\]

This system has the unique solution

\[ A_0 = \frac{6548}{16875}, \quad A_1 = \frac{6736 i}{16875}, \quad A_2 = \frac{424}{1125} + \frac{332 i}{1125}, \quad A_3 = -\frac{19}{75} + \frac{8 i}{75}, \quad A_3 = -\frac{2}{15} - \frac{1}{15} i. \]
Then the particular solution of the original equation,

\[ y_p = \text{Im} \ z_p = \text{Im} (e^{i\lambda} P_3) = e^{-t} \cdot \text{Im} \left( (\cos t + i \sin t)(A_0 + A_1 t + A_2 t^2 + A_3 t^3) \right) \]

\[ = e^{-t} \cos t \cdot \left( \frac{-6736}{16875} + \frac{332}{1125} t + \frac{8}{75} t^2 - \frac{1}{15} t^3 \right) + e^{-t} \sin t \cdot \left( \frac{6548}{16875} + \frac{424}{1125} t - \frac{19}{75} t^2 - \frac{2}{15} t^3 \right). \]

**Remark.** For comparison, the equation

\[ Ly = y''' + y'' + 3y' - 5y = t^3 e^{-t} \sin 2t \]

has a particular solution

\[ y_p = e^{-t} \cos 2t \cdot \left( -\frac{15}{512} - \frac{15}{256} t^2 - \frac{1}{64} t^3 + \frac{1}{64} t^4 \right) + e^{-t} \sin 2t \cdot \left( \frac{15}{512} - \frac{3}{256} t - \frac{3}{64} t^3 - \frac{1}{64} t^4 \right). \]

**#2.** Use Laplace transforms to solve the initial value problem

\[ y^{(4)} + 4y = t^2, \quad y(0) = y'(0) = y''(0) = y'''(0) = 0. \]

**Solution.** The Laplace transform \( Y = \mathcal{L} \{ y \} \) satisfies the equality

\[ (s^4 + 4) \cdot Y(s) = \mathcal{L} \{ t^2 \} = \frac{2}{s^3} \quad \implies \quad Y(s) = \frac{2}{s^3(s^4 + 4)}. \]

It is convenient to represent \( Y(s) \) as

\[ Y(s) = sZ(s), \quad \text{where} \quad Z(s) := \frac{2}{s^3(s^4 + 4)} = \frac{1}{2} \left( \frac{1}{s^3} - \frac{1}{s^4 + 4} \right). \]  \( (6) \)

The equation \( s^4 + 4 = 0 \) has roots \( s_k \), which can be determined from

\[ s_k^4 = -4 = 4 \cdot e^{i(-\pi + 2k\pi)} \quad \implies \quad s_k := \sqrt{2} \cdot e^{i(2k-1)\pi/4}, \quad k = 1, 2, 3, 4, \]

i.e. \( s_1 := 1 + i, \ s_2 := -1 + i, \ s_3 := -1 - i, \ s_4 := 1 - i. \) Then

\[ \frac{1}{s^4 + 4} = \sum_{k=1}^{4} \frac{A_k}{s - s_k}, \quad \text{where} \quad A_k := \lim_{s \to s_k} \frac{s - s_k}{s^4 + 4} = \frac{1}{4s_k^3} = \frac{s_k}{4s_k^4} = \frac{1}{16}. \]

Now from (6) it follows

\[ Y(s) = sZ(s) = \frac{1}{2s^3} + \frac{1}{32} \sum_{k=1}^{4} \frac{s_k s}{s - s_k}. \]

Since \( \sum s_k = 0 \), we can write, grouping together the complex conjugate term corresponding to pairs \( k = 1, 4 \) and \( k = 2, 3 \):

\[ \sum_{k=1}^{4} \frac{s_k s}{s - s_k} = \sum_{k=1}^{4} \left( \frac{s_k + \frac{s_k^2}{s - s_k}}{s - s_k} \right) = \sum_{k=1}^{4} \frac{s_k^2}{s - s_k} = \frac{2i}{s - 1 - i} + \frac{2i}{s + 1 + i} - \frac{2i}{s - 1 + i} = \frac{-4}{(s - 1)^2 + 1} + \frac{4}{(s + 1)^2 + 1}. \]

Finally,

\[ Y(s) = \frac{1}{2s^3} + \frac{1}{8} \cdot \left[ \frac{1}{(s + 1)^2 + 1} - \frac{1}{(s - 1)^2 + 1} \right], \]

and its inverse Laplace transform,

\[ y(t) = \mathcal{L}^{-1} \{ Y \} (t) = \frac{t^2}{4} + \frac{1}{8} \cdot (e^{-t} + e^t) \sin t = \frac{1}{4} \cdot (t^2 - \sinh t \cdot \sin t). \]
Indeed, replacing $y$ here corresponds to Problem 28 (c) in the 10th edition. It is treated there using the power series.\[\text{Remark.}\] This problem is contained in Section 6.2, Problem 26 in the textbook (11th edition), which approximately we conclude using the general fact (see Remark below).

**Solution.** Denote $Y(s) := \mathcal{L}\{y(t)\}(t)$. Then
\[
\mathcal{L}\{y'\} = sY - y(0) = sY - 1, \quad \mathcal{L}\{y''\} = s^2Y - s,
\]
\[
\mathcal{L}\{ty\} = -\frac{d}{ds}\mathcal{L}\{y\} = -Y', \quad \mathcal{L}\{t^2y\} = -\frac{d}{ds}\mathcal{L}\{y''\} = -(s^2Y-s)' = -s^2Y' - 2sY + 1.
\]
From the given equation it follows
\[
0 = \mathcal{L}\{ty'' + y' + ty\} = (-s^2Y' - 2sY + 1) + (sY - 1) - Y' = -(s^2 + 1)Y' - sY,
\]
\[
\frac{dY}{Y} = -\frac{ds}{s^2 + 1}, \quad \ln|Y| = \frac{1}{2}\ln(s^2 + 1) + c_0, \quad Y(s) = \frac{C}{\sqrt{s^2 + 1}}.
\]
Using the general fact (see Remark below)
\[
y(0) = \lim_{s \to \infty} sY(s), \quad (7)
\]
we conclude
\[
1 = y(0) = \lim_{s \to \infty} \frac{Cs}{\sqrt{s^2 + 1}} = \lim_{s \to \infty} \frac{C}{\sqrt{1 + s^{-2}}} = C, \quad \text{and} \quad Y(s) = \frac{1}{\sqrt{s^2 + 1}}.
\]
**Remark.** This problem is contained in Section 6.2, Problem 26 in the textbook (11th edition), which approximately corresponds to Problem 28 (c) in the 10th edition. It is treated there using the power series
\[
y(t) = \sum_{k=0}^{\infty} c_k t^k, \quad \text{which corresponds to} \quad Y(s) = \sum_{k=0}^{\infty} c_k \frac{s^k}{k!}.
\]
Here $C = c_0 = y(0) = 1$ is the coefficient of $s^{-1}$ in the expansion of $Y(s) = Cs^{-1} \cdot (1 + s^{-2})^{-1/2}$. This requires justification of term-by-term application of the Laplace transform to the series. Actually, the property (7) is more elementary and uses only the continuity: $y(t) \to y(0)$ at $t \to 0+$, and the standard growth condition
\[
|y(t)| \leq Me^{ct} \quad \text{for} \quad t \geq 0, \quad \text{with positive constants} \quad M \quad \text{and} \quad c. \quad (8)
\]
Indeed, replacing $y(t)$ by $y(t) - y(0)$, we can assume that $y(t) \to 0 = y(0)$ as $t \to 0+$. This means that for an arbitrary small $\varepsilon > 0$, there exists $\delta >$, such that $|y(t)| \leq \varepsilon$ on the interval $[0, \delta]$. We can write
\[
Y(s) = I_1 + I_2, \quad \text{where} \quad I_1 := \int_{0}^{\delta} se^{-st} f(t) dt, \quad I_2 := \int_{\delta}^{\infty} se^{-st} f(t) dt.
\]
Here

\[ |I_1| \leq \int_0^\delta se^{-st}|f(t)| \, dt \leq \varepsilon \int_0^\delta se^{-st} \, dt < \varepsilon, \]

\[ |I_2| \leq \int_\delta^\infty se^{-st}|f(t)| \, dt \leq Ms \int_\delta^\infty e^{-(s-c)t} \, dt = \frac{Ms}{s-c} \cdot e^{-(s-c)\delta} \to 0 \quad \text{as} \quad s \to \infty. \]

Then

\[ \lim_{s \to +\infty} |sY(s)| \leq \varepsilon, \]

and since the left hand side does not depend on \( \varepsilon \), we get

\[ sY(s) \to y(0) = 0 \quad \text{as} \quad s \to +\infty. \]

**#5.** Use Laplace transforms to solve the integral equation

\[ \int_0^t y(x)e^{xt} \, dx = y(t) - e^t. \]

**Solution.** This is the convolution \( y \ast e^t = y(t) - e^t \). The Laplace transform \( Y(s) := \mathcal{L}\{y(t)\}(s) \) satisfies

\[ Y(s) \cdot \frac{1}{s-1} = Y(s) - \frac{1}{s-1} \quad \implies \quad Y(s) = \frac{1}{s-2} \quad \implies \quad y(t) = e^{2t}. \]

**#6.** Show that \( f_1(t) := \int_0^\infty \frac{e^{-xt} \, dx}{x^2 + 1} \equiv f_2(t) := \int_0^\infty \frac{\sin(xt) \, dx}{x + 1} \) for \( t > 0 \).

**Proof.** Note that if \( F := \mathcal{L}\{f\} \), and \( G := \mathcal{L}\{g\} \), then by changing the order of integration, we get the following Parseval’s identity:

\[
\int_0^\infty F(x)g(x) \, dx = \int_0^\infty \int_0^\infty e^{-xy} f(y)g(x) \, dx \, dy = \int_0^\infty f(y)G(y) \, dy. \tag{9}
\]

In our example,

\[ F(x) := \frac{1}{x^2 + 1}, \quad g(x) := e^{-tx}, \quad f(y) := \sin y, \quad G(y) := \frac{1}{y + t}. \]

Then

\[ f_1(t) = \int_0^\infty F(x)g(x) \, dx = \int_0^\infty f(y)G(y) \, dy = \int_0^\infty \frac{\sin y}{y + t} \, dy. \]

After substitution \( y = xt \in \) the last integral, we get the desired equality \( f_1(t) = f_2(t) \).

**2nd way.** The Laplace transforms of \( f_1 \) and \( f_2 \) are

\[
F_1(s) = \int_0^\infty \frac{dx}{(x^2 + 1)(x + s)}, \quad F_2(s) = \int_0^\infty \frac{x \, dx}{(x + 1)(x^2 + s^2)}. \]

Using substitution \( x = sy^{-1} \) with \( s > 0 \), we reduce \( F_1(s) \) to

\[
F_1(s) = \int_0^\infty \frac{sy^{-2} \, dy}{(s^2 y^{-2} + 1)(sy^{-1} + s)} = \int_0^\infty \frac{y \, dy}{(y + 1)(y^2 + s^2)} = F_2(s). \]

By uniqueness, \( f_1(t) = f_2(t) \).