#1. Let $K$ be a nonempty compact set in a metric space $(X,d)$. Show that there are points $x_0, y_0 \in K$, such that

$$ \text{diam } K := \sup \{ d(x,y) : x,y \in K \} = d(x_0,y_0). $$

#2. Let $\overset{\circ}{E}$ denote the set of all interior points of a set $E$ in a metric space $(X,d)$. Show that

$$ (\overset{\circ}{E})^c = E^c. $$

#3. The boundary of a subset $E$ of a metric space $X$,

$$ \partial E := \{ p \in X : \forall r > 0 \text{ both } E \cap B_r(p) \text{ and } E^c \cap B_r(p) \text{ are nonempty} \}, $$

where $B_r(p) := \{ q \in X : d(p,q) < r \}$. Show that

$$ \partial E = \overline{E} \setminus \overset{\circ}{E} = \overline{E} \cap E^c. $$

In the remaining two problems, you have to use Definition 3.1 at the very beginning of Chapter 3.

#4. Let $X$ be a metric space of bounded sequences $x = \{x_1,x_2,\ldots,x_n,\ldots\}$ with the distance

$$ d(x,y) := \sup_n |x_n - y_n| \text{ for } x = \{x_n\}, \ y = \{y_n\}. $$

Show that the set

$$ K := \left\{ x = \{x_n\} \in X : |x_n| \leq c_n = \text{const for all } n = 1,2,\ldots \right\} $$

is compact in $X$ if and only if $c_n \to 0$ as $n \to \infty$.

#5. If $a_1 = 1$, and

$$ a_{n+1} = \frac{1}{1 + a_n} \text{ for } n = 1,2,\ldots, $$

prove that the sequence $\{a_n\}$ converges and find its limit. You can use without proof the fact that

$$ \lim_{n \to \infty} q^n = 0 \text{ if } |q| < 1. $$

*Hint.* Consider the differences $\varepsilon_n := a_n - L$, where $L$ is a positive solution of equation $L = \frac{1}{1+L}$. 

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Fall 2016. Math 5615H.  
Homework 6 (due on Wednesday, November 2).

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