
I.14. (4 points). From \(|z| = 1\) it follows \(z \bar{z} = |z|^2 = 1\), and

\[|1 + z|^2 + |1 - z|^2 = (1 + 2 \Re(z + z\bar{z}) + (1 - 2 \Re(z + z\bar{z}) = 4.\]

II.2. (5 points). Let \(F\) be a field. Show that there exist not more that two different solutions of the equation \(x \cdot x = 1\). Is it possible that there is only one solution to this equation?

**Solution.** If \(x \cdot x = 1\), then \((x - 1)(x + 1) = x^2 - 1 = 0\). Then \(x = 1\) or \(x = -1\). The solution is unique if these two values coincide, i.e. \(1 + 1 = 0\). This is possible, for example, in the simplest case \(F := \{0, 1\}\).

II.3. (4 points). Show that for any two rational numbers \(p_1 < p_2\) there exists an irrational number \(x\) such that \(p_1 < r < p_2\).

**Solution.** We have \(p_1 < x := p_1 + (p_2 - p_1)/\sqrt{2} < p_2\). If \(x\) is rational, then \(\sqrt{2} = (x - p_1)/(p_2 - p_1)\) is rational, which is not true. Hence \(x\) is irrational.

II.4. (7 points). Let \(A := \{a_1, a_2, \ldots\}\) be a set of real numbers defined as follows:

\[a_1 = 1, \quad \text{and} \quad a_{k+1} = 1 + \sqrt{a_k} \quad \text{for} \quad k = 1, 2, \ldots.\]

Find \(\sup A\).

**Solution.** By induction, \(1 \leq a_k < 4\) for all \(k \geq 1\), therefore, \(\exists L := \sup A \in [1, 4]\). Since \(a_{k+1} = 1 + \sqrt{a_k} \leq 1 + \sqrt{L}\) for all \(k\), we have \(L = \sup \{a_{k+1}\} \leq 1 + \sqrt{L}\). On the other hand, from \(1 + \sqrt{a_k} = a_{k+1} \leq L\) it follows \(1 + \sqrt{L} \leq L\), because in an equivalent form, \(a_k = (a_{k+1} - 1)^2 \leq (L - 1)^2\) implies \(L := \sup \{a_k\} \leq (L - 1)^2\). Therefore, \(1 + \sqrt{L} = L \in [1, 4]\), and \(L = (3 + \sqrt{5})/2\).