2. Let \( x \) any \( y \) be vectors in \( \mathbb{R}^k \). Show that \( y \) is uniquely represented in the form
\[
y = a + b, \quad \text{where} \quad a, b \in \mathbb{R}^k \quad \text{satisfy} \quad a = \alpha x \quad \text{for some real} \quad \alpha, \quad \text{and} \quad b \cdot x = 0.
\]
Verify this fact for \( x = (1, 1, 1) \) and \( y = (1, 2, 3) \) in \( \mathbb{R}^3 \).

**Proof.** Consider separately two cases (i) \( x = 0 \) and (ii) \( x \neq 0 \). In the case (i) we must have \( a = \alpha x = 0 \) and \( b = y \). In the case (ii), from \( y = a + b \) with \( a = \alpha x \) and \( b \cdot x = 0 \) it follows
\[
x \cdot y = x \cdot (a + b) = \alpha x \cdot x + x \cdot b = \alpha |x|^2, \quad \text{so that} \quad \alpha = \frac{x \cdot y}{|x|^2}, \quad b = y - \alpha x
\]
are defined uniquely, and \( b \) indeed satisfies \( b \cdot x = (y - \alpha x) \cdot x = y \cdot x - \alpha |x|^2 = 0 \).

If \( x = (1, 1, 1) \) and \( y = (1, 2, 3) \), then
\[
x \cdot y = 6, \quad |x|^2 = 3, \quad \alpha = 2, \quad a := \alpha x = (2, 2, 2), \quad b := y - \alpha x = (-1, 0, 1),
\]
so that \( b \cdot x = 0 \).

3. Let \( A \) be a nonempty set in \( \mathbb{R}^k \). Define
\[
d(x) := \inf \{|x - a| : a \in A\}.
\]
Show that
\[
|d(x) - d(y)| \leq |x - y| \quad \text{for all} \quad x, y \in \mathbb{R}^k.
\]

**Proof.** By definition of \( d(x) \) and Theorem 1.37(f),
\[
d(x) \leq |x - a| \leq |x - y| + |y - a| \quad \forall x, y \in \mathbb{R}^k, \quad \text{and} \quad a \in A.
\]
Then \( d(x) - |x - y| \leq d(y) \), i.e. \( d(x) - d(y) \leq |x - y| \). By symmetry, we also have \( d(y) - d(x) \leq |x - y| \), and the desired inequality follows.

4. Show that \( (1 + h)^n \geq 1 + nh \) for all natural \( n \) and real \( h \) such that \( |h| \leq 1 \). Using this result, show that
\[
a_1 \leq a_2 \leq a_3 \leq \ldots, \quad \text{where} \quad a_n = \left(1 + \frac{1}{n}\right)^n.
\]

**Proof.** The inequality \( (1 + h)^n \geq 1 + nh \) is obviously true for \( n = 1 \). If it is true for some \( n = k \geq 1 \), then multiplying its both sides by \( 1 + h \geq 0 \), we get
\[
(1 + h)^{k+1} \geq (1 + kh)(1 + h) = 1 + (k + 1)h + kh^2 \geq 1 + (k + 1)h,
\]
i.e. it is also true for \( n = k + 1 \). By induction, it is true for all natural \( n \). Using this inequality with \( n \geq 2 \) and \( h = -n^{-2} \), we obtain
\[
\frac{a_n}{a_{n-1}} = \left(\frac{n+1}{n}\right)^n \left(\frac{n-1}{n}\right)^{n-1} = \left(1 - \frac{1}{n^2}\right)^n \cdot \frac{n}{n-1} \geq \left(1 - \frac{n}{n^2}\right) \cdot \frac{n}{n-1} = 1,
\]
i.e. \( a_{n-1} \leq a_n \) for all \( n \geq 2 \).

5. Evaluate
\[
\cos \frac{\pi}{15} \cdot \cos \frac{2\pi}{15} \cdot \cos \frac{4\pi}{15} \cdot \cos \frac{8\pi}{15}.
\]

**Proof.** Denote \( \theta = \frac{\pi}{15} \). Then
\[
\cos \theta \cdot \cos 2\theta \cdot \cos 4\theta \cdot \cos 8\theta = \frac{\sin 2\theta}{2 \sin \theta} \cdot \frac{\sin 4\theta}{2 \sin 2\theta} \cdot \frac{\sin 8\theta}{2 \sin 4\theta} = \frac{\sin 16\theta}{16 \sin \theta} = \frac{1}{16}.
\]