#1. Using partial-fraction decomposition

\[
\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2},
\]

show that the series

\[
\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}
\]

covers and find its sum.

**Solution.** We have

\[
\frac{1}{n(n+1)(n+2)} = \frac{1}{2} \left( \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} \right)
\]

Then the \(n\)th partial sum

\[
S_n = \sum_{k=1}^{n} \left( \frac{1}{2} \left( \frac{1}{k} - \frac{1}{k+1} + \frac{1}{k+2} \right) \right)
\]

\[
= \left( \frac{1}{2} \frac{1}{1} - \frac{1}{2} + \frac{1}{3} \right) + \left( \frac{1}{2} \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \right) + \ldots
\]

\[
+ \left( \frac{1}{2} \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n+1} \right) + \left( \frac{1}{2} \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} \right)
\]

\[
= \frac{1}{4} + \frac{-1/2}{n+1} + \frac{1/2}{n+2} \rightarrow S = \frac{1}{4} \text{ as } n \to \infty.
\]

In other words, the sum of this series \(S = 1/4\).

#2. Find a constant \(c_0 > 0\) satisfying the following properties.

(i). The convergence of every series \(\sum a_n\) with \(a_n \geq 0\) implies the convergence of

\[
\sum_{n=1}^{\infty} b_n := \sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n^c}
\]

for each \(c > c_0\).

(ii). There is a convergent series \(\sum a_n\) with \(a_n \geq 0\), such that the series \(\sum b_n\) diverges if \(c = c_0\).

**Solution.** (i). By the elementary inequality \(ab \leq \left( a^2 + b^2 \right) / 2\) with \(a = \sqrt{a_n}\) and \(b = n^{-c}\), we have

\[
b_n := \frac{\sqrt{a_n}}{n^c} \leq \frac{1}{2} (a_n + n^{-2c})
\]

If \(c > 1/2\), then the series \(\sum n^{-2c}\) converges by Theorem 3.28. Using also Theorem 3.25(a), one can see that the convergence of \(\sum a_n\) with \(a_n \geq 0\) implies the convergence of \(\sum b_n\).

(ii). On the other hand, if \(c = 1/2\), then one can take \(a_1 = 0\) and

\[
a_n := \frac{1}{n \ln n}, \quad b_n := \frac{\sqrt{a_n}}{n^c} = \frac{1}{n \ln n} \text{ for } n = 2, 3, \ldots.
\]

By Theorem 3.29, \(\sum a_n\) converges, and \(\sum b_n\) diverges.

Therefore, both statements (i) and (ii) are true with \(c_0 := 1/2\).

#3. The sequence \(\{a_n\}\) is defined by

\[
a_1 = 0, \quad \text{and} \quad a_{n+1} = 2^{a_n/2} \quad \text{for} \quad n = 1, 2, 3, \ldots.
\]

Prove that \(\{a_n\}\) is convergent and find its limit. You can use without proof the fact that the function \(y = f(x) = a^x\), where \(a > 0, a \neq 1\), is strictly convex, i.e.

\[
f(tx_1 + (1-t)x_2) < t f(x_1) + (1-t) f(x_2)
\]

for all \(x_1 < x_2\) and \(0 < t < 1\).
Solution. By induction, \( a_1 = 0 < a_2 = 1 < a_3 < \cdots < a_n < \cdots < 2 \) for all \( n = 1, 2, 3, \ldots \). By Theorem 3.14 and its proof, there exists \( L = \lim a_n = \sup \{ a_n \} \leq 2 \). Moreover,
\[
\begin{align*}
a_{n+1} = 2^{a_n / 2} & \implies a_{n+1} \leq 2^{L / 2}, \quad L \geq 2^{a_n / 2} \quad \text{for} \quad n = 1, 2, 3, \ldots \\
& \implies L \leq 2^{L / 2}, \quad L \geq 2^{L / 2} \implies L = 2^{L / 2} \leq 2.
\end{align*}
\]

We claim that \( L = 2 \), i.e. the case \( L = 2^{L / 2} < 2 \) is impossible. Indeed, if this is the case, then the equation \( x = 2^{x / 2} \) has at least 3 distinct solutions \( x_1 = L < x_0 = 2 < x_2 = 4 \). In other words, the graph of a strictly convex function \( y = f(x) := 2^{x / 2} \) intersects the line \( y = x \) at 3 distinct points, which is impossible. Formally, we can write
\[
x_0 = tx_1 + (1 - t)x_2, \quad \text{where} \quad t := \frac{x_2 - x_0}{x_2 - x_1} \quad \text{if} \quad x_0 \in (x_1, x_2).
\]

By the property of strict convexity, the desired contradiction follows:
\[
x_0 = f(x_0) = f((tx_1 + (1 - t)x_2) < t f(x_1) + (1 - t) f(x_2) = tx_1 + (1 - t)x_2 = x_0.
\]

\#4. (Exercise 6(a–c) on p.78). Investigate the behavior (convergence or divergence) of \( \sum a_n \) if
\[
\text{(a)} \quad a_n = \sqrt{n+1} - \sqrt{n}; \quad \text{(b)} \quad a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}; \quad \text{(c)} \quad a_n = (\sqrt{n} - 1)^n.
\]

Solution. (a). Since the partial sums
\[
s_n := \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left( \sqrt{k+1} - \sqrt{k} \right) = \sqrt{n+1} - 1 \to +\infty,
\]
the series \( \sum a_n \) diverges.

(b). Note that
\[
0 < a_n := \frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{1}{n \cdot (\sqrt{n+1} + \sqrt{n})} < c_n := \frac{1}{n^{3/2}}.
\]

By Theorem 3.28, the series \( \sum c_n \) converges. Then by Theorem 3.25(a), \( \sum a_n \) also converges.

(c). We know that \( \sqrt{n} - 1 \to 0 \) (Theorem 3.20(c)). Therefore, there exists a natural \( N_0 \) such that \( 0 < \sqrt{n} - 1 < 1/2 \) for all \( n \geq N_0 \). Then
\[
0 < a_n := (\sqrt{n} - 1)^n \leq 2^{-n} \quad \text{for all} \quad n \geq N_0,
\]
and \( \sum a_n \) converges by Theorems 3.20(e) and 3.25(a).

\#5. Using the fact that
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \text{find} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}.
\]

Solution. We have
\[
\begin{align*}
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} &= \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \\
&= \left( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \right) - 2 \cdot \left( \frac{1}{2^2} + \frac{1}{4^2} + \cdots \right) \\
&= \left( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right) - \frac{1}{2} \cdot \left( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right) \\
&= \left( 1 - \frac{1}{2} \right) \cdot \left( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right) = \frac{\pi^2}{12}.
\end{align*}
\]