Appendix A. Compactness in Metric Spaces.

In the textbook by Walter Rudin, Principles of Mathematical Analysis, 3rd edition, 1976, the compactness is defined by the following **Heine-Borel property**:

A subset $K$ in a metric space $X$ is **compact**, if

$$K \subset \bigcup_{\alpha} G_\alpha \quad \text{with open} \quad G_\alpha \quad \implies \quad K \subset \bigcup_{j=1}^{n} G_{\alpha_j} \quad \text{for some finite subfamily of} \quad \{G_\alpha\}.$$

**Theorem A1.** The Heine-Borel property is equivalent to the following **Weierstrass property**: every infinite subset $E \subset K$ has a limit point in $K$.

**Proof.** The Weierstrass property follows from the Heine-Borel property by Theorem 2.37 in the textbook. It suffices to show that if the Heine-Borel property fails, then the Weierstrass property fails as well. Therefore, suppose that

$$K \subset \bigcup_{\alpha} G_\alpha \quad \text{with open} \quad G_\alpha \quad \text{without a finite subcover}.$$ 

Since $Q_\alpha$ are open,

$$\forall p \in K, \quad \exists r(p) \in \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots \right\} \quad \text{such that} \quad p \in B_{r(p)}(p) \subset G_\alpha \quad \text{for some} \quad \alpha.$$ 

Pick $p_1 \in K$ with the maximal possible $r(p_1)$, and then by induction, for $k = 2, 3, \ldots$,

$$p_k \in K \setminus \left(\bigcup_{j=1}^{k-1} B_{r(p_j)}(p_j)\right)$$

with the maximal possible $r(p_k)$. The subset $E := \{p_1, p_2, \ldots, p_n, \ldots\} \subset K$ is infinite, because otherwise $\{B_{r(p_j)}(p_j) \subset G_\alpha\}$ would be a finite subcover of $K$, and correspondingly, $\{G_\alpha\}$ would also be a finite subcover of $K$.

We claim that the set $E$ has no limit point. Suppose otherwise: let $p_0 \in K$ be a limit point of $E$. Consider two possible cases.

(i) $p_0 \in B_{r(p_k)}(p_k)$ for some $k$. Since all these balls are open, we have $p_0 \in B_{r}(p_0) \subset B_{r(p_k)}(p_k)$ for some $\varepsilon > 0$. By constructions, all the point $p_j$ with $j \geq k + 1$ lie outside of $B_{r(p_k)}(p_k)$, hence $B_{\varepsilon}(p_0)$ can only contain a finite number of point $p_j$, and by Theorem 2.20, $p_0$ cannot be a limit point of $E$.

(ii) $p_0 \notin B_{r(p_k)}(p_k)$ for all $k$. Once again by construction, we must have $0 < r(p_0) \leq r(p_k)$ for all $k$. Then $d(p_0, p_k) \geq r(p_k) \geq r(p_0) > 0$, i.e. $p_k \notin B_{r(p_0)}(p_0)$ for all $k$, and $p_0$ is not a limit point of $E$.

In any case, the Weierstrass property fails for the infinite set $E$, which completes the proof. \(\square\)

This theorem can be re-formulated in the following form. In one direction, this statement is contained in Theorem 3.6(a) in the textbook.

**Theorem A2.** A subset $K$ of a metric space $(X, d)$ is compact if and only if every sequence $\{p_n\} \subset K$ contains a convergent subsequence in $K$. 


We give one more convenient criterion of compactness.

**Definition A3.** A subset $E$ of a metric space $(X,d)$ is totally bounded if $\forall \varepsilon > 0$, there exists a finite subset $\{p_1, p_2, \ldots, p_n\} \subset E$ such that

$$ E \subset \bigcup_{j=1}^{n} B_{\varepsilon}(p_j). $$

**Theorem A4.** A subset $K$ of a metric space $(X,d)$ is compact if and only if it is (i) complete and (ii) totally bounded.

**Proof.** Let $K$ be a compact subset of $X$. The completeness of $K$ is contained in Theorem 3.11(b) in the textbook. Alternatively, one can use the above Theorem A1 in the following way. Let $\{p_n\}$ be a Cauchy sequence in $K$. If the set $E := \{p_n\} \subset K$ is finite, then we obviously have $p = p_{n_1} = p_{n_2} = \cdots$ for some sequence of natural indices $n_1 < n_2 < \cdots$. In this case trivially $p_{n_j} \to p$ as $j \to \infty$. If the set $E$ is infinite, then by Theorem A1 it has a limit point $p \in K$. By Theorem 3.2(d), the set $E$ contains a convergent subsequence $p_{n_j} \to p \in K$ as $j \to \infty$, so that this property holds true in any case. Next, since $\{p_n\}$ is a Cauchy sequence, $\forall \varepsilon > 0, \exists N = N(\varepsilon) > 0$ such that $d(p_n, p_m) < \varepsilon$ for all $m, n \geq N$. Then

$$ d(p_n, p) \leq d(p_n, p_{n_j}) + d(p_{n_j}, p) < \varepsilon + d(p_{n_j}, p), \quad \forall n, n_j \geq N. $$

By taking limit as $j \to \infty$, we get $d(p_n, p) \leq \varepsilon, \forall n \geq N$. This implies that $\{p_n\}$ converges to $p$, so that $K$ is complete.

Compact sets $K$ must be totally bounded, because otherwise we get infinite set $E := \{p_n\} \subset K$ with $d(p_j, p_k) \geq \varepsilon > 0, \forall j \neq k$. Then $E'$ is empty, in contradiction to Theorem A1.

Now suppose that $K$ is complete and totally bounded, and let $E$ be an infinite subset of $K$. Starting from $E_0 := E$ and using total boundedness, we can define a decreasing sequence of infinite sets

$$ E_n := E_{n-1} \cap B_{1/n}(p_n) \quad \text{for some distinct points} \quad p_n \in K, \quad n = 1, 2, \ldots. $$

By this construction, we have

$$ d(p_n, p_m) < \frac{1}{n}, \quad \forall m > n. $$

This implies that $\{p_n\}$ is a Cauchy sequence. By completeness, $p_n \to p \in E'$, so that $E'$ is nonempty. By Theorem A1, the subset $K$ is compact in $(X,d)$. \qed