
**Problem 1.** (12 points). For an arbitrary set \( E \subseteq \mathbb{R}^2 \), consider its projection \( E_1 \) on the \( x \)-axis, which is defined as follows:

\[
x \in E_1 \iff \exists y \in \mathbb{R} \text{ such that } (x, y) \in E.
\]

Either prove or give a counterexample for each of the following statements.

(a). If \( E \) is open in \( \mathbb{R}^2 \), then \( E_1 \) is open in \( \mathbb{R} \).

(b). If \( E \) is closed in \( \mathbb{R}^2 \), then \( E_1 \) is closed in \( \mathbb{R} \).

(c). If \( E \) is compact in \( \mathbb{R}^2 \), then \( E_1 \) is compact in \( \mathbb{R} \).

**Solution.** (a). True. For an arbitrary \( x \in E_1 \), there exists \( y \in \mathbb{R} \) such that \((x, y) \in E\). Since \( E \) is open in \( \mathbb{R}^2 \), there exists a disk \( B_r(P) \subseteq E \) of some radius \( r > 0 \). Then its projection \((x-r, x+r) \subseteq E_1\), so that \( E_1 \) is open.

(b). False. The graph of function \( y = 1/x \), i.e. the set \( E := \{(x, y) \in \mathbb{R}^2 : x > 0, y = 1/x\} \), is closed in \( \mathbb{R}^2 \). However, its projection \( E_1 = \{x \in \mathbb{R} : x > 0\} \) is not closed in \( \mathbb{R} \).

(c). True. For any sequence \( \{x_n\} \subseteq E_1 \subseteq \mathbb{R} \), there exists a sequence \( \{P_n\} := \{(x_n, y_n)\} \subseteq E \subseteq \mathbb{R}^2 \). Since \( E \) is compact in \( \mathbb{R}^2 \), there exists a convergent subsequence \( P_{n_k} := (x_{n_k}, y_{n_k}) \to P_0 := (x_0, y_0) \in E \) as \( k \to \infty \).

Then \( x_{n_k} \to x_0 \in E_1 \), so that \( E_1 \) is compact.

**Problem 2.** (15 points). Calculate

(a) \( \lim_{n \to \infty} (\sqrt{n^2 + n} - n) \);  
(b) \( \lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n \).

**Solution.** (a). Note that

\[
\sqrt{n^2 + n} - n = \frac{(\sqrt{n^2 + n} - n)(\sqrt{n^2 + n} + n)}{\sqrt{n^2 + n} + n} = \frac{n}{\sqrt{n^2 + n} + n} = \frac{1}{\sqrt{1 + n^{-1}} + 1}.
\]

At this point, we only have arithmetic properties of limits, which do not include \( \lim \sqrt{a_n} = \sqrt{\lim a_n} \). Instead of it, we write

\[
\left| \sqrt{1 + n^{-1}} - 1 \right| = \frac{n^{-1}}{\sqrt{1 + n^{-1}} + 1} \leq \frac{1}{n} \to 0 \text{ as } n \to +\infty,
\]

and finally get

\[
\lim_{n \to \infty} (\sqrt{n^2 + n} - n) = \lim_{n \to \infty} \frac{1}{\sqrt{1 + n^{-1}} + 1} = \frac{1}{\lim_{n \to \infty} \sqrt{1 + n^{-1}} + 1} = \frac{1}{2}.
\]

(b). Substitute \( n = m + 1 \). Then

\[
\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n \to \infty} \left(\frac{n}{n-1}\right)^n = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{m+1} = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m \cdot \lim_{m \to \infty} \left(1 + \frac{1}{m}\right) = e \cdot 1 = e.
\]
Problem 3. (15 points). Find the limit
\[ L = \lim_{n \to \infty} \frac{n}{\sqrt{n!}}, \]
or show that it does not exist.

This is Problem #2 in Homework #8.

Problem 4. (18 points). Suppose \( c_n \geq 0 \) for all \( n = 1, 2, 3, \ldots \).

What can be said about convergence or divergence of the series

(a) \( \sum_{n=1}^{\infty} a_n \) and (b) \( \sum_{n=1}^{\infty} b_n \), where \( a_n := \frac{c_n}{n^2 + c_n^2} \), \( b_n := \frac{c_n}{n^3 + c_n^3} \)?

Since these series converge for \( c_n = 0 \), the possible answers are: (i) converges for arbitrary sequence \( c_n \geq 0 \), or (ii) diverges for some sequence \( c_n \geq 0 \).

Provide the proof for one of these options separately in cases (a) and (b).

Solution. (a). The series \( \sum a_n \) diverges if \( c_n = n \) by Theorem 3.28, because

\[ a_n = \frac{c_n}{n^2 + c_n^2} = \frac{n}{n^2 + n} = \frac{1}{n + 1}. \]

(b). Note that

\[ b_n := \frac{c_n}{n^3 + c_n^3} \leq \frac{n}{n^3 + c_n^3} \leq \frac{1}{n^2} \quad \text{if} \quad 0 \leq c_n \leq n, \]

\[ \leq \frac{c_n}{c_n^3} < \frac{1}{n^2} \quad \text{if} \quad c_n > n. \]

In any case, we have \( 0 \leq b_n \leq n^{-2} \), and since \( \sum n^{-2} \) converges by Theorem 3.28, the series \( \sum b_n \) converges for arbitrary \( c_n \geq 0 \). \( \square \)