Appendix B. Extension of Continuous Functions

Let \( f(x) \) be a continuous function on a compact set \( K \subset \mathbb{R}^d \). Then it is uniformly continuous on \( K \), i.e. its **modulus of continuity**

\[
\omega(\rho) := \sup \left\{ |f(x) - f(y)| : \quad x, y \in K, \quad |x - y| \leq \rho \right\} \searrow 0 \quad \text{as} \quad \rho \searrow 0.
\]  

(1)

**Lemma 1.** If \( K \) is convex, then \( \omega(\rho) \) is subadditive, i.e

\[
\omega(\rho_1 + \rho_2) \leq \omega(\rho_1) + \omega(\rho_2) \quad \text{for} \quad \rho_1, \rho_2 \geq 0.
\]  

(2)

**Proof.** For arbitrary \( x, y \in K \) with \( |x - y| \leq \rho_1 + \rho_2 \), the segment \([x, y]\) lies in \( K \), and there is a point \( z \in [x, y] \) such that \(|x - z| \leq \rho_1, |y - z| \leq \rho_2\). Therefore,

\[
|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| \leq \omega(\rho_1) + \omega(\rho_2),
\]

and (2) follows. \( \square \)

**Remark 2.** The property (2) fails in general if \( K \) is not convex. In polar coordinates \( x = r \cos \theta, y = r \sin \theta \), an easy example is

\[
f(x, y) := \theta \quad \text{on} \quad K := \{1 \leq r \leq 2, \quad \varepsilon \leq \theta \leq 2\pi - \varepsilon\} \quad \text{with a small} \quad \varepsilon > 0.
\]

**Lemma 3.** For any continuous function \( f \) on a compact \( K \subset \mathbb{R}^d \) with the modulus of continuity \( \omega(\rho) \) in (1), the function

\[
\overline{\omega}(\rho) := \sup_{s \geq 1} \frac{\omega(ps)}{s}, \quad \rho \geq 0,
\]

satisfies the properties:

(i) \( \overline{\omega}(\rho) \geq \omega(\rho) \),

(ii) \( \overline{\omega}(\rho) \searrow 0 \) as \( \rho \searrow 0 \),

(iii) \( \overline{\omega}(\rho) \) is subadditive, i.e. it satisfies (2).

**Proof.** (i) is obvious.

(ii) Since \( \omega(\rho) \) is non-decreasing, the function \( \overline{\omega}(\rho) \) is also non-decreasing for \( \rho \geq 0 \). We also have

\[
\omega(\rho) \leq C_0 := 2 \sup_K |f|, \quad \text{and} \quad \overline{\omega}(\rho) \leq C_0.
\]

For an arbitrary \( A > 1 \), we can write

\[
\overline{\omega}(\rho) = \max \left\{ \sup_{A \geq s \geq 1} \frac{\omega(ps)}{s}, \quad \sup_{s \geq A} \frac{\omega(ps)}{s} \right\} \leq \max \left\{ \omega(A\rho), \frac{C_0}{A} \right\}.
\]

This implies

\[
\limsup_{\rho \to 0^+} \overline{\omega}(\rho) \leq \frac{C_0}{A},
\]

and since \( A > 1 \) can be chosen arbitrarily large, we have \( \overline{\omega}(\rho) \searrow 0 \) as \( \rho \searrow 0 \).
(iii) We can write
\[ \bar{\omega} := \sup_{s \geq 1} \frac{\omega(\rho s)}{s} = \rho \cdot \sup_{t \geq \rho} \frac{\omega(t)}{t} \quad \text{for} \quad \rho > 0. \]

Therefore,
\[
\bar{\omega}(\rho_1 + \rho_2) = (\rho_1 + \rho_2) \cdot \sup_{t \geq \rho_1 + \rho_2} \frac{\omega(t)}{t} \\
\leq \rho_1 \cdot \sup_{t \geq \rho_1} \frac{\omega(t)}{t} + \rho_2 \cdot \sup_{t \geq \rho_2} \frac{\omega(t)}{t} \\
= \bar{\omega}(\rho_1) + \bar{\omega}(\rho_2). 
\]

\[\square\]

**Theorem 4.** Let \( f \) be a continuous function on a compact \( K \subset \mathbb{R}^d \). Then the function
\[
F(x) := \inf_{y \in K} \left[ f(y) + \bar{\omega}(|x - y|) \right], \quad x \in \mathbb{R}^d, \tag{4}
\]

where \( \bar{\omega}(\rho) \) is defined in (3), satisfies the properties:

(i) \( F \equiv f \) on \( K \),

(ii) \( F \) provides a continuous extension of \( f \) from \( K \) to \( \mathbb{R}^d \), and

\[ |F(x_1) - F(x_2)| \leq \bar{\omega}(|x_1 - x_2|), \quad \forall x_1, x_2 \in \mathbb{R}^d. \tag{5} \]

**Proof.** (i) Note that \( \forall x, y \in K \), we have
\[
f(x) \leq f(y) + \omega(|x - y|) \leq f(y) + \bar{\omega}(|x - y|),
\]
with the equality at \( y = x \). Therefore, \( f(x) = F(x) \), \( \forall x \in K \).

(ii) By monotonicity and subadditivity of \( \bar{\omega}(\rho) \), we have \( \forall x_1, x_2, y \in K \):
\[
\bar{\omega}(|x_1 - y|) \leq \bar{\omega}(|x_1 - x_2| + |x_2 - y|) \leq \bar{\omega}(|x_1 - x_2|) + \bar{\omega}(|x_2 - y|).
\]

Hence the function \( F(x) \) in (4) satisfies
\[
F(x_1) \leq \bar{\omega}(|x_1 - x_2|) + F(x_2).
\]

Interchanging \( x_1 \) and \( x_2 \), we get the desired property (5).

\[\square\]