Math 5616H: Extra Credit Problem.

#4 (Ch. 8: #18). Find out, for each of two functions

(i) \( f(x) := x^3 - \sin^2 x \cdot \tan x, \)
(ii) \( g(x) := 2x^2 - \sin^2 x - x \tan x, \)

whether it is positive or negative for all \( x \in (0, \pi/2), \) or whether it changes sign.

**Solution.** We will show that we have \( f(x) < 0, g(x) < 0 \) for all \( x \in (0, \pi/2). \)

(i) The desired inequality is equivalent to

\[
\left( \frac{\sin x}{x} \right)^3 > \cos x, \quad \text{or equivalently, to} \quad \sin^6 x > x^6 \cos^2 x = x^6(1 - \sin^2 x).
\]

Denote \( y := \sin x, \) so that \( x := \arcsin y. \) Now we need to prove that

\( x := \arcsin y < y \cdot (1 - y^2)^{-1/6} \quad \text{for} \quad y \in (0, 1). \)

In turn, since the both sides vanish at \( y = 0, \) it suffices to get the inequality for their derivatives:

\[
(1 - y^2)^{-1/2} < (1 - y^2)^{-1/6} + y \cdot \left( -\frac{1}{6} \right) \cdot (1 - y^2)^{-7/6} \cdot (-2y),
\]

or, multiplying by \( (1 - y^2)^{-7/6}, \)

\[
(1 - y^2)^{2/3} < (1 - y^2) + \frac{y^2}{3} = 1 - \frac{2y^2}{3}.
\]

In the formula

\[
(1 + t)^a = 1 + \frac{a}{1!} \cdot t + \frac{a(a - 1)}{2!} \cdot t^2 + \cdots \quad \text{with} \quad t := -y^2, \quad a := \frac{2}{3},
\]

all the terms starting with \( t^2 \) are strictly negative, which implies the desired inequality. Alternatively, one can use an elementary inequality

\[
(1 - 3s)^2 < (1 - 2s)^3 \quad \text{with} \quad s := y^{2/3} \in \left(0, \frac{1}{3}\right).
\]

(i) Since \( f(x) < 0, \) we have \( \tan x \cdot \sin^2 x > x^3. \) Therefore,

\[
(x \tan x + \sin^2 x)^2 = (x \tan x - \sin^2 x)^2 + 4x \tan x \cdot \sin^2 x \geq 4x \tan x \cdot \sin^2 x > 4x^4.
\]

Then \( x \tan x + \sin^2 x > 2x^2, \) and \( g(x) < 0 \) for all \( x \in (0, \pi/2). \)