#1 (Ch. 8: #1). Define
\[
f(x) = \begin{cases} 
  e^{-1/x^2} & (x \neq 0), \\
  0 & (x = 0). 
\end{cases}
\]
Prove that \( f \) has derivatives of all orders at \( x = 0 \), and that \( f^{(n)}(0) = 0 \) for \( n = 1, 2, 3, \ldots \).

**Hint.** First prove by induction that
\[
f^{(n)}(x) = f(x) \cdot P_n(x) \cdot x^{-3n} \quad \text{for all} \quad x \neq 0 \quad \text{and} \quad n = 1, 2, 3, \ldots,
\]
where \( P_n(x) \) is a polynomial of some degree. Then note that from Taylor’s expansion
\[
e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} < \frac{t^k}{k!} \quad \text{for all} \quad k = 1, 2, \ldots \quad \text{with} \quad t = x^{-2} > 0
\]
it follows
\[
f(x) = e^{-t} < \frac{k!}{t^k} = k! \cdot x^{2k} \quad \text{for all} \quad k = 1, 2, \ldots.
\]

#2 (Ch. 8: #5). Find the following limits
\[
(a) \lim_{x \to 0} \frac{e - (1 + x)^{1/x}}{x}, \quad (b) \lim_{n \to \infty} \frac{n}{\ln n} (n^{1/n} - 1).
\]

**Hint.** Write
\[
(1 + x)^{1/x} = e^{\ln(1+x)/x}, \quad n^{1/n} = e^{(\ln n)/n}
\]
and use the limits
\[
\lim_{\alpha \to 0} \frac{e^{\alpha} - 1}{\alpha} = 1, \quad \lim_{\alpha \to 0} \frac{\ln(1 + \alpha)}{\alpha} = 1.
\]

#3 (Ch. 8: #6(a)). Suppose \( f(x)f(y) = f(x + y) \) for all real \( x \) and \( y \). Assuming that \( f \) is differentiable and not zero, prove that
\[
f(x) = e^{cx},
\]
where \( c \) is a constant.

#4 (Ch. 8: #7). If \( 0 < x < \pi/2 \), prove that
\[
\frac{2}{\pi} < \frac{\sin x}{x} < 1.
\]