#1 (Ch. 8: #1). Define

\[ f(x) = \begin{cases} e^{-1/x^2} & (x \neq 0), \\ 0 & (x = 0). \end{cases} \]

Prove that \( f \) has derivatives of all orders at \( x = 0 \), and that \( f^{(n)}(0) = 0 \) for \( n = 1, 2, 3, \ldots \).

**Hint.** First prove by induction that

\[ f^{(n)}(x) = f(x) \cdot P_n(x) \cdot x^{-3n} \quad \text{for all} \quad x \neq 0 \quad \text{and} \quad n = 1, 2, 3, \ldots, \]

where \( P_n(x) \) is a polynomial of some degree. Then note that from Taylor’s expansion

\[ e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} > \frac{t^k}{k!} \quad \text{for all} \quad k = 1, 2, \ldots \quad \text{with} \quad t = x^{-2} > 0 \]

it follows

\[ f(x) = e^{-t} < \frac{k!}{t^k} = k! \cdot x^{2k} \quad \text{for all} \quad k = 1, 2, \ldots. \]

#2 (Ch. 8: #6(a)). Suppose \( f(x)f(y) = f(x+y) \) for all real \( x \) and \( y \). Assuming that \( f \) is differentiable and not zero, prove that

\[ f(x) = e^{cx}, \]

where \( c \) is a constant.

#3. Same problem under assumption that \( f(x) \) is a continuous function, a priori not necessarily differentiable.

#4 (Ch. 8: #7). If \( 0 < x < \pi/2 \), prove that

\[ \frac{2}{\pi} < \frac{\sin x}{x} < 1. \]