50 points are divided between 4 problems.

#1. (10 points.) Prove that the series
\[
\sum_{n=1}^{\infty} \frac{\sin n\theta}{n^p}
\]
converges for all real \( \theta \) and \( p > 0 \).

This problem deals with review of the previous material, namely Theorem 3.44, in combination with Euler’s formulas.

#2. (10 points.) Find the coefficients \( a_n \) in the power series
\[
f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{for} \quad |x| < R \quad \text{if} \quad a_0 = 1 \quad \text{and} \quad f'(x) = -2xf(x).
\]

#3. (15 points.) Show that for all real \( x > 0 \),
\[
\begin{align*}
\text{(a) } & \quad x - \frac{x^2}{2} < \ln(1 + x) < x, \\
\text{(b) } & \quad x - \frac{x^3}{6} < \sin x < x, \\
\text{(c) } & \quad \left(1 + \frac{1}{x}\right)^x < e < \left(1 + \frac{1}{x}\right)^{x+1}.
\end{align*}
\]

#4. (15 points.) Consider the sequence
\[
a_n = \sqrt{n} \cdot \left(\frac{n}{e}\right)^n.
\]

It is known that
\[
\sum_{k=0}^{2n} \binom{2n}{k} \cdot 2^{-2n} = 1, \quad \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \quad \text{and} \quad \lim_{n \to \infty} \frac{n!}{a_n} = C = \text{const} > 0.
\]

Use these relations to show that \( C = \sqrt{2\pi} \).