#1 (Ch. 5: #14). (i) Let $f$ be a differentiable real function defined in $(a,b)$. Prove that $f$ is convex if and only if $f'$ is monotonically increasing.

(ii) Assume next that $f''(x)$ exists for every $x \in (a,b)$, and prove that $f$ is convex if and only if $f''(x) \geq 0$ for all $x \in (a,b)$.

**Proof.** (i) Let $f$ be a convex function in $(a,b)$, i.e. according to the definition in problem #23 on p.101,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad \text{for all} \quad x, y \in (a,b) \quad \text{and} \quad 0 \leq \lambda \leq 1. \quad (1)$$

For fixed $a < x < y < b$, the function

$$g(\lambda) := f(\lambda x + (1 - \lambda)y) - \lambda f(x) - (1 - \lambda)f(y)$$

satisfies $g \leq 0$ on $[0,1]$, and $g(0) = g(1) = 0$. Then it attains its maximum on $[0,1]$ at the boundary points 0 and 1, which implies $g'(0) \leq 0 \leq g'(1)$. By the chain rule,

$$g'(0) = f'(y) \cdot (x - y) - f(x) + f(y) \leq g'(1) = f'(x) \cdot (x - y) - f(x) + f(y).$$

Since $x < y$, we have $x - y < 0$, $f'(x) \leq f'(y)$, and $f'$ is monotonically increasing.

Now suppose that $f'$ is monotonically increasing in $(a,b)$. Replacing $\lambda$ by $1 - \lambda$ if necessary, it suffices to verify the convexity condition (1) for fixed $a < x < y < b$. Take a linear function $l(t)$ such that $l(x) = f(x)$ and $l(y) = f(y)$, namely

$$l(t) := f(t) + k \cdot (t - x), \quad \text{where} \quad k := \frac{f(y) - f(x)}{y - x}. $$

Then $h(t) := f(t) - l(t)$ satisfies $h(x) = h(y) = 0$, and $h' = f' + k$ is monotonically increasing. Note that the equality (1) turns into equality for linear functions. Since $f = h + l$, it suffices to verify (1) for the function $h$ with fixed $x < y$. This reduces the proof of (1) to

$$h(t) \leq 0 \quad \text{for all} \quad t \in [x,y]. \quad (2)$$

By the mean value theorem 5.10, there is a point $z \in (x,y)$ with $h'(z) = 0$. Since $h'$ is monotone, we have $h' \leq 0$ on $[x,z]$, and $h' \geq 0$ on $[z,y]$. By Theorem 5.11, $h$ decreases on $[x,z]$ and increases on $[z,y]$. Together with the boundary conditions $h(x) = h(y) = 0$, this yields the desired inequality (2).

(ii) By the previous part (i), we only need to show that $f'$ is monotonically increasing in $(a,b) \iff f'' \geq 0$ in $(a,b)$. If $f'$ is increasing in $(a,b)$, then $f'(x + h) \geq f'(x)$ for all $a < x < x + h < b$, and

$$f''(x) = \lim_{h \to 0^+} f'(x + h) - f'(x) \geq 0. $$

On the other hand, if $f'' \geq 0$ in $(a,b)$, then $f'$ in increasing by Theorem 5.11(a) applied to $f'$.
#2 (Ch. 5: #22(a,b)). Suppose \( f(t) \) is a real function on \((-\infty, \infty)\). Call \( x \) a fixed point of \( f \) if \( f(x) = x \).

(a) If \( f \) is differentiable and \( f'(t) \neq 1 \) for every real \( t \), prove that \( f \) has at most one fixed point.

(b) Show that the function \( f(t) := t + (1 + e^t)^{-1} \) has no fixed point, although \( 0 < f'(t) < 1 \) for all real \( t \).

Proof. (a) If \( a < b \) are two distinct fixed points, then by the mean value theorem (Theorem 5.10),
\[
b - a = f(b) - f(a) = (b - a) \cdot f'(x)
\]
for some \( x \in (a, b) \). After cancellation by \( b - a \neq 0 \), we get \( f'(x) = 1 \), in contradiction to our assumption \( f' \neq 1 \). Therefore, \( f \) cannot have more than one fixed point.

(b) The equality \( f(x) = x \) is impossible, because \((1 + e^x)^{-1} \neq 0 \). Therefore, \( f \) cannot have more than one fixed point. We also have
\[
0 < f'(t) = 1 - \frac{e^t}{(1 + e^t)^2} < 1 \quad \text{for all real } t.
\]

#3 (Ch. 6: #10(a,b)). Let \( p \) and \( q \) be positive real numbers such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Prove the following statements.

(a) If \( u \geq 0 \) and \( v \geq 0 \), then
\[
uv \leq \frac{u^p}{p} + \frac{v^q}{q}.
\]
Equality holds if and only if \( u^p = v^q \).

(b) If \( f \) and \( g \) are Riemann integrable on \([a, b]\), \( f \geq 0 \), \( g \geq 0 \), and
\[
\int_a^b f^p \, dx = 1 = \int_a^b g^q \, dx,
\]
then
\[
\int_a^b fg \, dx \leq 1.
\]

Proof. (a) Obviously, it suffices to consider the case \( u > 0 \), \( v > 0 \). Denote
\[
a := u^p > 0, \quad b := v^q > 0, \quad \text{and} \quad \lambda := p^{-1} \in (0, 1).
\]
Then the desired inequality is reduced to
\[
a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b \quad \text{for all} \quad a > 0, \ b > 0, \ \lambda \in (0, 1).
\]
Since \( \ln x \) is a strictly monotone function in \((0, +\infty)\), the last inequality is in turn equivalent to
\[
\lambda \ln a + (1-\lambda) \ln b \leq \ln (\lambda a + (1-\lambda)b) \quad \text{for all} \quad a > 0, \ b > 0, \ \lambda \in (0, 1).
\](3)

Note that \((\ln x)'' = -x^{-2} < 0\). Then \( f(x) := -\ln x \) satisfies \( f''(x) > 0 \) in \((0, +\infty)\), hence it is convex in \((0, +\infty)\), and (3) follows from (1).
Finally, the equality in (1) for some \( x < y \) and \( 0 < \lambda < 1 \) implies that the function \( h(t) := f(t) - l(t) \) in the proof of the above Problem 1(i) has at least 3 zeroes \( x < t_0 < y \). Then \( h'(t) \) has at least 2 zeroes \( t_1 \in (x, t_0) \) and \( t_2 \in (t_0, y) \). In turn, this implies that \( h''(z) = f''(z) = 0 \) for some \( z \in (t_1, t_2) \subset (x, y) \). This is impossible in our case, because \( f'' \neq 0 \). Therefore, the equality in (3) is only possible if and only if \( a = b \), i.e. \( u^p = v^q \).

(b) Using the part (a) with \( u = f \) and \( v = g \), we get

\[
\int_a^b fg \, d\alpha \leq \frac{1}{p} \int_a^b f^p \, d\alpha + \frac{1}{q} \int_a^b g^q \, d\alpha = \frac{1}{p} + \frac{1}{q} = 1.
\]

#4 (Ch. 6: #15). Suppose \( f \) is a real, continuously differentiable function on \([a, b]\), such that \( f(a) = f(b) = 0 \), and

\[
\int_a^b f^2(x) \, dx = 1.
\]

Prove that

\[
\int_a^b xf(x)f'(x) \, dx = -\frac{1}{2},
\]

and that

\[
\int_a^b \left[f'(x)\right]^2 \, dx \cdot \int_a^b x^2 f^2(x) \, dx > \frac{1}{4}.
\]

**Proof.** Using integration by parts (Theorem 6.22) and boundary conditions \( f(a) = f(b) = 0 \), we obtain

\[
1 = \int_a^b f^2(x) \, dx = -\int_a^b xd(f^2(x)) = -2 \int_a^b xf(x)f'(x) \, dx,
\]

and (4) follows. For the proof of (5), we will use the Schwarz inequality

\[
\left| \int_a^b f_1 f_2 \, dx \right| \leq \|f_1\|_2 \cdot \|f_2\|_2,
\]

where \( \|f\|_2 := \left( \int_a^b f^2 \, dx \right)^{1/2} \) in a stronger form: the equality here holds if and only if the functions \( f_1 \) and \( f_2 \) are linearly dependent. This is a special case of Problem 10(c), p.139, when \( p = q = 2 \), and some additional arguments are needed in order to handle the case of the equality. Multiplying \( f_1 \) and \( f_2 \) by appropriate constants, we can reduce the proof to the case \( \|f_1\|_2 = \|f_2\|_2 = 1 \). In this case,

\[
0 \leq \int_a^b (f_1 \pm f_2)^2 \, dx = \int_a^b f_1^2 \, dx + \int_a^b f_2^2 \, dx \pm 2 \int_a^b f_1 f_2 \, dx = 2 \pm 2 \int_a^b f_1 f_2 \, dx.
\]

This implies (6) with equality if and only if \( f_1 = f_2 \) or \( f_1 = -f_2 \), depending on the sign of the integral in the left hand side.

In our situation, we can take \( f_1(x) := f'(x) \) and \( f_2(x) := xf(x) \). Then (6) is reduced to

\[
\frac{1}{2} \leq \|f'_1\|_2 \cdot \|xf\|_2.
\]

We will show that the equality here is impossible. Indeed, if this is the case, then by the previous argument \( f'(x) = cxf(x) \) with a constant \( c \). This implies \( |f'(x)| \leq A |f(x)| \) on \([a, b] \) with \( A = \text{const} > 0 \).
By Problem #20 in Ch. 5 (Problem #3 in Homework #1), we must have $f \equiv 0$ on $[a, b]$, which is impossible by our assumptions. Therefore, we have the strict inequality in (7), which is equivalent to (5).

#5. Let $f$ be a real smooth function on $\mathbb{R}$ such that $f(x) \equiv 0$ for $|x| \geq 1$. For fixed constant $\alpha := 1 + \beta \in (1, 2)$, introduce

$$F_1 := \sup_{x \in \mathbb{R}, h > 0} \left| \frac{f'(x + h) - f'(x)}{h^\beta} \right|, \quad F_2 := \sup_{x \in \mathbb{R}, h > 0} \left| \frac{f(x + h) - 2f(x) + f(x - h)}{h^\alpha} \right|.$$ 

Show that if $F_1 < \infty$, then $F_1 \leq NF_2$, where $N$ is a constant depending only on $\alpha$.

**Remark.** The assumption $f(x) \equiv 0$ for $|x| \geq 1$ is actually not needed.

**Proof.** Using notation $T_h f(x) := f(x + h) - f(x)$, as in the solution of Problem #5 in HW #1, we can write

$$F_1 = \sup_{x \in \mathbb{R}, h > 0} h^{-\beta}|T_h f'(x)|, \quad F_2 = \sup_{x \in \mathbb{R}, h > 0} h^{-\alpha}|T_h^2 f(x)|.$$

Since $F_1 < \infty$, one can fix $x_0 \in \mathbb{R}$ and $H > 0$, such that

$$\frac{1}{2} : F_1 \leq H^{-\beta} \cdot |T_H f'(x_0)|.$$

Set $h := m^{-1}H > 0$, where $m$ is a large natural number to be specified below. By Theorem 5.10, we have

$$T_h f(x_0) = h \cdot f'(y_0), \quad T_h f(x_0 + H) = h \cdot f'(z_0), \quad \text{where} \quad |x_0 - y_0| < h, \quad |x_0 + H - z_0| < h.$$

By definition of $F_1$,

$$|h f'(x_0) - T_h f(x_0)| = h \cdot |f'(x_0) - f'(y_0)| \leq hF_1 \cdot |x_0 - y_0|^\beta \leq h^\alpha F_1,$$

and similarly, $|h f'(x_0 + H) - T_h f(x_0 + H)| \leq h^\alpha F_1$. Since $H = mh$ and $\alpha = 1 + \beta$, we obtain

$$\frac{1}{2} : F_1 \leq H^{-\beta} \cdot |f'(x_0 + H) - f'(x_0)| = m^{-\beta} h^{-\alpha} \cdot |h f'(x_0 + H) - h f'(x_0)|.$$

Here

$$|h f'(x_0 + H) - h f'(x_0)| \leq |h f'(x_0 + H) - T_h f(x_0 + H)| + |h f'(x_0) - T_h f(x_0)|$$

$$+ |T_h f(x_0 + H) - T_h f(x_0)| \leq 2h^\alpha F_1 + |T_h f(x_0 + H) - T_h f(x_0)|.$$

Now from the previous relation it follows

$$\frac{1}{2} : F_1 \leq 2m^{-\beta} F_1 + m^{-\beta} h^{-\alpha} \cdot |T_h f(x_0 + H) - T_h f(x_0)|.$$

Finally, choose a natural $m$ depending only on $\alpha = 1 + \beta$, such that $2m^{-\beta} \leq \frac{1}{4}$. After cancellation, we obtain

$$F_1 \leq N_1 h^{-\alpha} \cdot |T_h f(x_0 + mh) - T_h f(x_0)|, \quad \text{where} \quad N_1 = N_1(\alpha) = 4m^{-\beta} = \text{const}.$$

It remains to notice that by definition of $F_2$,

$$|T_h f(x_0 + mh) - T_h f(x_0)| = \left| \sum_{k=0}^{m-1} \left( T_h f(x_0 + kh + h) - T_h f(x_0 + kh) \right) \right| = \left| \sum_{k=0}^{m-1} T_h^2 f(x_0 + kh) \right| \leq m \cdot h^\alpha F_2.$$

Therefore, $F_1 \leq NF_2$ with $N := N_1 m$. \qed