Homework 5. Problems and Solutions.

#1. (10 points.) Prove that the series
\[
\sum_{n=1}^{\infty} \frac{\sin n\theta}{n^p}
\]
converges for all real \( \theta \) and \( p > 0 \).

This problem deals with review of the previous material, namely Theorem 3.44, in combination with Euler’s formulas.

Proof. If \( \theta = k\pi \) with an integer \( k \), then \( \sin n\theta = 0 \) for all \( n \), and this claim is obvious. Otherwise we have \( z := e^{i\theta} \neq 1 \), and by Theorem 3.44 with \( c_n := n^{-p} \setminus 0 \), the series
\[
\sum_{n=0}^{\infty} \frac{z^n}{n^p} = \sum_{n=0}^{\infty} \left( \frac{\cos n\theta}{n^p} + i \cdot \frac{\sin n\theta}{n^p} \right)
\]
converges. Therefore, its imaginary part also converges, which completes the proof. Note that the real part diverges for \( z = 1, \ 0 < p \leq 1 \).

#2. (10 points.) Find the coefficients \( a_n \) in the power series
\[
f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{for} \quad |x| < R \quad \text{if} \quad a_0 = 1 \quad \text{and} \quad f'(x) = -2xf(x).
\]

Solution. From the equalities \( f(0) = a_0 = 1 \) and \( f'(x) = -2xf(x) \) it follows
\[
(\ln f(x))' = -2x, \quad \ln f(x) = -x^2, \quad f(x) = e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!},
\]
which in turn implies that
\[
a_n = \frac{(-1)^k}{k!} \quad \text{for} \quad n = 2k, \quad \text{and} \quad a_n = 0 \quad \text{for} \quad n = 2k + 1.
\]

#3. (15 points.) Show that for all real \( x > 0 \),

(a) \( x - \frac{x^2}{2} < \ln(1 + x) < x \),

(b) \( x - \frac{x^3}{6} < \sin x < x \),

(c) \( \left(1 + \frac{1}{x}\right)^x < e < \left(1 + \frac{1}{x}\right)^{x+1} \).

Proof (a). These inequalities follow immediately by integration from 0 to \( x \) of all parts of the inequalities
\[
1 - t < \frac{1}{1+t} < 1 \quad \text{for} \quad t > 0.
\]
(b). We have
\[ -1 < -\cos t \quad \text{for} \quad 0 < t \neq 2n\pi, \quad n \in \mathbb{Z}. \]
By integrating, we get for \( x > 0 \):
\[
-x = \int_0^x (-1) \, dt < \int_0^x (-\cos t) \, dt = -\sin x \\
1 - \frac{x^2}{2} = 1 + \int_0^x (-t) \, dt < 1 + \int_0^x (-\sin t) \, dt = \cos x \leq 1 \\
x - \frac{x^3}{6} = \int_0^x (1 - \frac{t^2}{2}) \, dt < \int_0^x \cos t \, dt = \sin x < x.
\]
(c). By taking the logarithm of all parts, we see that these inequalities are equivalent to
\[
\frac{1}{1 + x} < \ln \left( 1 + \frac{1}{x} \right) < \frac{1}{x}, \quad x > 0.
\]
In turn, by substitution \( t = 1/x \), they are reduced to
\[
\frac{t}{1 + t} < \ln(1 + t) < 1, \quad t > 0.
\]
The second inequality is contained in (a), and the first one follows from
\[
\frac{t}{1 + t} = 1 - \frac{1}{1 + t} = \int_0^t \frac{ds}{(1 + s)^2} < \int_0^t \frac{ds}{1 + s} = \ln(1 + t), \quad t > 0.
\]

#4. (15 points.) Consider the sequence
\[
a_n = \sqrt{n} \cdot \left( \frac{n}{e} \right)^n.
\]
It is known that
\[
\sum_{k=0}^{2n} \binom{2n}{k} \cdot 2^{-2n} = 1, \quad \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \quad \text{and} \quad \lim_{n \to \infty} \frac{n!}{a_n} = C = \text{const} > 0.
\]
Use these relations to show that \( C = \sqrt{2\pi} \).

**Proof.** Throughout the proof, we will use the notation \( x_n \sim y_n \) if \( x_n/y_n \to 1 \) as \( n \to \infty \). In particular, we can rewrite one of our assumptions as
\[
n! \sim C a_n, \quad \text{where} \quad C = \text{const} > 0. \tag{1}
\]

First we show that the values of \( c_k := \binom{2n}{k} \cdot 2^{-2n} \) are “concentrated” near \( k = n \). For this purpose, consider the generating function
\[
g(t) := \sum_{k=0}^{2n} c_k t^k = (1 + t)^n 2^{-2n}, \quad \text{which satisfies} \quad \sum_{k=0}^{2n} c_k = g(0) = 1.
\]
Moreover,
\[ g'(t) = \sum_{k=0}^{2n} kc_k t^{k-1} = 2n(1 + t)^{2n-1}a^{2n}, \quad \sum_{k=0}^{2n} kc_k = g'(0) = n; \]
\[ g''(t) = \sum_{k=0}^{2n} k(k-1)c_k t^{k-1} = 2n(2n-1)(1 + t)^{2n-2}a^{2n}, \quad \sum_{k=0}^{2n} (k^2 - k)c_k = g''(0) = n^2 - \frac{n}{2}. \]

These relations imply
\[ \sum_{k=0}^{2n} (k - n)^2c_k = \sum_{k=0}^{2n} [(k^2 - k) - (2n - 1)k + n^2]c_k = \left(n^2 - \frac{n}{2}\right) - (2n - 1)n + n^2 = \frac{n}{2}. \]

Now we can restrict the summation to \( k \) satisfying
\[ |k - n| \leq n^{2/3}, \tag{2} \]
because
\[ \sum_{\{k: |k-n|>n^{2/3}\}} c_k \leq \sum_{k=0}^{2n} \left(\frac{k-n}{n^{2/3}}\right)^2 c_k = \frac{n}{2n^{4/3}} = \frac{1}{2n^{1/3}} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \]

From (1) and (2) it follows that
\[ c_k = \frac{(2n)! \cdot 2^{-2n}}{k!(2n-k)!} \sim \frac{C\sqrt{2n} \cdot 2^{-2n}}{C\sqrt{k} \cdot C\sqrt{2n-k}} \cdot \left(\frac{2n}{e}\right)^{2n} \left(\frac{e}{k}\right)^k \left(\frac{e}{2n-k}\right)^{2n-k} \sim \frac{\sqrt{2}}{C\sqrt{n}} \cdot A_k, \tag{3} \]
where
\[ A_k := \frac{n^{2n}}{k^k(2n-k)^{2n-k}}. \]

It is convenient to express \( A_k \) in terms of \( \alpha_k := \frac{k}{n} - 1 \) as follows:
\[
-\ln A_k = k \ln k + (2n - k) \ln(2n - k) - 2n \ln n = k \ln \frac{k}{n} + (2n - k) \ln \left(2 - \frac{k}{n}\right)
\[
= n \cdot \left[(1 + \alpha_k) \ln(1 + \alpha_k) + (1 - \alpha_k) \ln(1 - \alpha_k)\right].
\]

Since the last expression is an even function of \( \alpha_k \), so that its Taylor’s expansion does not contain odd powers of \( \alpha_k \), and by virtue of (2) \( n\alpha_k^2 \leq n^{-1/3} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \), we get a simple relation
\[
-\ln A_k = n \cdot \left[(1 + \alpha_k) \left(\alpha_k - \frac{\alpha_k^2}{2} + \cdots\right) - (1 - \alpha_k) \left(\alpha_k + \frac{\alpha_k^2}{2} + \cdots\right)\right] = n\alpha_k^2 + \varepsilon_k,
\]
where \( \varepsilon_k \) become uniformly small for large \( n \). Now (3) is reduced to a more precise relation
\[
c_k \sim \frac{\sqrt{2}}{C\sqrt{n}} \cdot e^{-n\alpha_k^2} = \frac{\sqrt{2}}{C\sqrt{n}} \cdot e^{-\frac{(k-n)^2}{n}} = \frac{\sqrt{2}}{C\sqrt{n}} \cdot e^{-x_k^2}, \quad \text{where} \quad x_k := \frac{k-n}{\sqrt{n}}.
\]

Combining together the previous relations, we get
\[
1 \sim \sum_{\{k: |k-n|\leq n^{2/3}\}} c_k \sim \frac{\sqrt{2}}{C} \cdot \sum_{\{k: |x_k|\leq n^{1/6}\}} \frac{e^{-x_k^2}}{\sqrt{n}}.
\]
Finally, it remains to note that the last sum is a Riemann sum with easily controllable terms, so that

\[ 1 = \frac{\sqrt{2}}{C} \cdot \int_{-\infty}^{\infty} e^{-x^2} \, dx = \frac{\sqrt{2\pi}}{C}, \quad \text{and} \quad C = \sqrt{2\pi}. \]

**Remark.** The equivalence \( n! \sim \sqrt{2\pi} \cdot a_n \) is known as J. Stirling’s formula (1730). Its modern proofs usually consist of two steps: (i) prove that \( n! \sim C \cdot a_n \) with \( C = \text{const} > 0 \), and then (ii) show that \( C = \sqrt{2\pi} \). It is possible to prove by very elementary means that

\[ n! = C a_n e^{\theta_n} = C \sqrt{n} \cdot \left( \frac{n}{e} \right)^n e^{\theta_n}, \quad \text{where} \quad \frac{1}{12n+1} < \theta_n < \frac{1}{12n} \]

(see [1], and also [2], Sec. II.9), and there are numerous ways to show that \( C = \sqrt{2\pi} \). According to [3], Chapter VIII, the equivalence \( n! \sim C \cdot a_n \) with undetermined constant \( C > 0 \) was known to A. de Moivre (1718), who later in 1738 gave full credit to Stirling, and also the complete proof of two-sided estimate for \( n! \) with \( 0 < \theta_n < 1/12n \) was first derived by an Italian mathematician Cesaro (1884). This two-sided estimate is very sharp: for \( n = 10 \) we have

\[ \sqrt{20\pi} \left( \frac{10}{e} \right)^{10} e^{1/121} \approx 3,628,560.1 < 10! = 3,628,800 < \sqrt{20\pi} \left( \frac{10}{e} \right)^{10} e^{1/120} \approx 3,628,810.1. \]

Our way to find value of \( C \) is probabilistic in nature and is closer to [2], Sec. VII.2. Namely, we deal with a particular case of a random variables \( X_m \) which have binomial distribution with parameters \( m \in \mathbb{N} \) and \( p \in [0, 1] \), when \( m = 2n \) and \( p = 1/2 \). In general case,

\[ c_k = P(X_m = k) = \binom{m}{k} p^k q^{m-k}, \quad q := 1 - p, \quad \text{for} \quad k = 0, 1, 2, \ldots, m. \]

We follow a standard way of computation of the **mathematical expectation** and **variance** of \( X_m \): 

\[ \mu := E(X_m) := \sum_{k=0}^{m} k c_k = mp, \quad \text{Var}(X_m) := E[(X_m - \mu)^2] := \sum_{k=0}^{m} [(k - \mu)^2] c_k = mpq, \]

and then using **Chebyshev’s inequality**

\[ P(|X_m - \mu| \geq c > 0) \leq \frac{\text{Var}(X_m)}{c^2}. \]

The remaining part partially follows the lines (with different normalization) of the classical de Moivre–Laplace limit theorem (see [2], Theorem VII.3.1).

Using the above relations, one can also prove the Weierstrass theorem by using **Bernstein’s polynomials**: if \( f(p) \) is a continuous function on \([0, 1]\), then

\[ \lim_{m \to \infty} \sum_{k=0}^{m} f \left( \frac{k}{m} \right) \binom{m}{k} p^k (1-p)^{m-k} = \lim_{m \to \infty} Ef \left( \frac{X_m}{m} \right) = f(p) \quad \text{uniformly on} \quad [0, 1]. \]

**REFERENCES**