Problem 1. Consider Legendre polynomials

\[ L_n(x) := c_n \cdot [(x^2 - 1)^n]^{(n)}, \quad \text{where} \quad c_n := \frac{1}{2^n \cdot n!}, \quad n = 0, 1, 2, \ldots \tag{1} \]

(a). Show that

\[ \int_{-1}^{1} L_m L_n \, dx = 0 \quad \text{for} \quad m \neq n. \]

(b). Prove the identities

\[ (n + 1)L_{n+1}(x) = (2n + 1)x L_n(x) - n L_{n-1}(x) \quad \text{for} \quad n = 1, 2, \ldots. \tag{2} \]

**Proof.** (a). Without loss of generality, we can assume that \( m > n \). Note that the points \( x = \pm 1 \) are zeros of order \( m \) of \( (x^2 - 1)^m = (x - 1)^m(x + 1)^m \). Therefore, all derivatives \( [(x^2 - 1)^n]^{(k)} \) for \( k < m - 1 \) vanish at \( x = \pm 1 \). Moreover, since \( L_n \) is a polynomial of degree \( n < m \), we also have \( L_n^{(m)} = 0 \). Integrating by parts \( m \) times, we get the desired orthogonality of \( L_m \) and \( L_n \) for \( m \neq n \):

\[
\int_{-1}^{1} L_m L_n \, dx = c_m \int_{-1}^{1} [(x^2 - 1)^m]^{(m)} L_n \, dx = (-1)^m c_m \int_{-1}^{1} (x^2 - 1)^m L_n^{(m)} \, dx = 0.
\]

(b). We will derive this relation from a few basic properties of polynomials \( L_n \).

(i). \( L_m \) is orthogonal to every polynomial \( P_n \) of degree \( n < m \).

Indeed, the previous argument works with \( P_n \) in place of \( L_n \).

(ii). \( xL_n \) is orthogonal to every polynomial \( P_k \) of degree \( k \leq n - 2 \).

Indeed, the orthogonality condition

\[ \int_{-1}^{1} xL_n P_k \, dx = 0 \]

is the same as orthogonality of \( L_n \) to polynomial \( xP_k \) of degree \( k + 1 \leq n - 1 \), which is true because of (i).

(iii). \( L_n \) contain only even powers of \( x \) for even \( n \), and only odd powers of \( x \) for odd \( n \).

This follows from the fact that \((x^2 - 1)^n\) contains only even powers of \( x \). The properties (i)–(iii) imply

(iv). \( xL_n = aL_{n+1} + bL_{n-1} \), where \( a \) and \( b \) are constants (depending on \( n \)).

In order to find these constants, we need two more relations.

(v). \( L_n(1) = 1 \) for every \( n \), hence \( a + b = 1 \).

Indeed, using Leibniz formula for derivative of product with \( f := (x + 1)^n \) and \( g := (x - 1)^n \), we get (because only \( k = 0 \) produces nonzero \( g^{(n-k)}(1) \)):

\[ L_n(1) = c_n (fg)^{(n)}(1) = c_n \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(1) g^{(n-k)}(1) = c_n 2^n n! = 1. \]

(vi).

\[ L_n(x) = \frac{(2n)!}{2^n (n!)^2} \cdot x^n + \text{lower powers of } x. \]

Here the coefficient of \( x^n \) comes from \((x^2)^{(n)} = 2n(2n-1) \cdots (n+1)x^n = \frac{(2n)!}{n!} \cdot x^n.\]
Comparing the coefficients of \(x^{n+1}\) in (iv), we finally obtain
\[
\frac{(2n)!}{2^n(n!)^2} = a \cdot \frac{(2n + 2)!}{2^{n+1}((n + 1)!)^2}, \quad \text{and} \quad a = \frac{2(n + 1)^2}{(2n + 2)(2n + 1)} = \frac{n + 1}{2n + 1}, \quad b = 1 - a = \frac{n}{2n + 1}.
\]
The relation (iv) with these values of \(a\) and \(b\) is equivalent to (2).

**Remark.** The equality (1) is known as Rodrigues’ formula and (2) – as Bonnet’s recursion formula. Polynomials \(L_n\) can be defined in a number of different ways, e.g. by orthogonalization of \(1, x, x^2, \ldots\) subject to restriction \(L_n(1) = 1\).

**Problem 2.** Find the minimum of
\[
f(c_0, c_1, c_2) := \int_{-1}^{1} (x^3 - c_0 - c_1 x - c_2 x^2)^2 \, dx \quad \text{over} \quad c_0, c_1, c_2 \in \mathbb{R}.
\]

**Solution.** Using (2), one can easily get:
\[
L_0 = 1, \quad L_1 = x, \quad L_2 = \frac{1}{2} \cdot (3x^2 - 1), \quad L_3 = \frac{1}{2} \cdot (5x^3 - 3x).
\]
We can write
\[
x^3 - c_0 - c_1 x - c_2 x^2 = \frac{2}{5} \cdot L_3 + P_2,
\]
where \(P_2\) is a polynomial of degree at most 2. Since \(L_3\) is orthogonal to \(P_2\), the integral
\[
\int_{-1}^{1} (x^3 - c_0 - c_1 x - c_2 x^2)^2 \, dx = \int_{-1}^{1} \left(\frac{2}{5} \cdot L_3\right)^2 \, dx + \int_{-1}^{1} P_2^2 \, dx
\]
is minimal when \(P_2 = 0\), i.e. \(c_0 = 0, c_1 = 3/5, c_2 = 0\). The corresponding minimal value is
\[
\int_{-1}^{1} \left(\frac{2}{5} \cdot L_3\right)^2 \, dx = 2 \int_{0}^{1} \left(x^3 - \frac{3}{5} \cdot x\right)^2 \, dx = 2 \int_{0}^{1} \left(x^6 - \frac{6}{5} \cdot x^4 + \frac{9}{25} \cdot x^2\right) \, dx = 2 \cdot \left(\frac{1}{7} - \frac{6}{25} + \frac{3}{25}\right) = 8/175.
\]

**Remark.** The previous calculation can be made even shorter if we use (for \(n = 3\)) the equality
\[
\int_{-1}^{1} L_n^2 \, dx = \frac{2}{2n + 1}.
\]
Indeed, as in the previous solution of Problem 1(a),
\[
\int_{-1}^{1} L_n^2 \, dx = (-1)^n c_n^2 \int_{-1}^{1} (x^2 - 1)^n L_n^{(n)} \, dx = c_n^2 (2n)! \int_{-1}^{1} (1 - x^2)^n \, dx.
\]
By substitution
\[
x = -1 + 2t, \quad 1 + x = 2t, \quad 1 - x = 2(1 - t), \quad dx = 2dt \quad \text{for} \quad 0 \leq t \leq 1,
\]

2
the last integral is reduced to
\[ \int_{-1}^{1} (1 - x^2)^n \, dx = 2^{2n+1} \int_{0}^{1} t^n (1 - t)^n \, dt = 2^{2n+1} B(n + 1, n + 1) = \frac{2^{2n+1} \Gamma^2(n + 1)}{\Gamma(2n + 2)} = \frac{2^{2n+1} (n!)^2}{(2n + 1)!}. \]

After cancelation, we arrive at (3).

**Problem 3.** Find the limit
\[ \lim_{n \to \infty} n \cdot \left[ \left(1 + \frac{1}{n}\right)^n - e \right]. \]

**Solution.** We can write
\[
\begin{align*}
n \cdot \left[ \left(1 + \frac{1}{n}\right)^n - e \right] &= n \cdot \left[ e^{n \ln \left(1 + \frac{1}{n}\right)} - e \right] = n \cdot \left[ e^{n \left(\frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)\right)} - e \right] \\
&= en \cdot \left[ e^{-\frac{1}{2n} + o\left(\frac{1}{n}\right)} - 1 \right] = en \cdot \left( -\frac{1}{2n} + o\left(\frac{1}{n}\right) \right) \to -\frac{e}{2} \quad \text{as} \quad n \to \infty.
\end{align*}
\]

Therefore, the answer is \(-e/2\).

**Problem 4.** Consider the functions
\[
\Phi(x) := \int_{0}^{x} e^{-t^2} \, dt, \quad F(x) := \Phi^2(x), \quad \text{and} \quad G(x) := \int_{0}^{1} \frac{e^{-\left(1+t^2\right)x^2} \, dt}{1 + t^2}.
\]

(a). Show that \(F + G \equiv C = \text{const.}\)

(b). Find the value of \(C\).

**Solution.** (a). It suffices to show that \(F' + G' = 0\). We have
\[
F'(x) = 2\Phi' \Phi(x) = 2e^{-x^2} \int_{0}^{x} e^{-t^2} \, dt.
\]

Further, since the integral function in \(G(x)\) is smooth with respect to \(x, t\), we get
\[
G'(x) = \int_{0}^{1} \frac{\partial}{\partial x} \left[ e^{-\left(1+t^2\right)x^2} \right] \, dt = -2x \int_{0}^{1} e^{-\left(1+t^2\right)x^2} \, dt.
\]

By substitution \(y = xt, dy = xdt\), we arrive at the desired claim
\[
G'(x) = -2e^{-x^2} \int_{0}^{x} e^{-y^2} \, dy = -F'(x), \quad (F + G)'(x) = 0, \quad \text{and} \quad F + G \equiv C = \text{const.}
\]

(b). We just take \(x = 0\):
\[
C = F(0) + G(0) = 0 + \int_{0}^{1} \frac{dt}{1 + t^2} = \arctan 1 = \frac{\pi}{4}.
\]
Remark. This problem provides an alternative proof of a famous equality

\[ \int_{-\infty}^{\infty} e^{-t^2} dt = 2 \cdot \lim_{x \to +\infty} \Phi(x) = 2 \cdot \lim_{x \to +\infty} \sqrt{C - G(x)} = 2\sqrt{C} = \sqrt{\pi}. \]

Problem 5. Find the generating function

\[ F(t, x) := \sum_{n=0}^{\infty} \frac{t^n H_n(x)}{n!} \quad \text{for the Hermite polynomials} \quad H_n(x) := (-1)^n e^{x^2} \left(e^{-x^2}\right)^{(n)}. \]

Solution. Using Taylor’s expansion

\[ f(x + h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} h^n \quad \text{with} \quad f(x) = e^{-x^2}, \ h = -t, \]

we get

\[ F(t, x) = e^{x^2} \cdot \frac{(e^{-x^2})^{(n)}}{n!} \cdot (-t)^n = e^{x^2} \cdot e^{-(x-t)^2} = e^{2tx-t^2}. \]

Problem 6. Simplify the expression

\[ f(x) = \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{2^2 x^3}{1+x^4} + \cdots + \frac{2^n x^{2n-1}}{1+x^{2^n}} + \cdots \quad \text{for} \quad |x| < 1. \]

Solution. One of possible ways is to find the coefficients \(a_n\) in the power series

\[ f(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{1}{1+x} + 2x \cdot f(x^2) = \sum_{n=0}^{\infty} (-1)^n x^n + 2 \sum_{k=0}^{\infty} a_k x^{2k+1}. \]

Comparing the coefficients, we see that \(a_{2k} = 1\) for \(k = 0, 1, 2, \ldots\), and also \(a_{2k+1} = -1 + 2a_k = 1\) by easy induction. Thus \(a_n = 1\) for all \(n\), and

\[ f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for} \quad |x| < 1. \]

Remark. This equality follows by differentiation of the series

\[ \ln(1+x) + \ln(1+x^2) + \ln(1+x^4) + \cdots + \ln(1+x^{2^n}) + \cdots = -\ln(1-x). \]

In turn, the last equality follows from

\[
\begin{align*}
\ln(1-x) + \ln(1+x) + \ln(1+x^2) + \ln(1+x^4) + \cdots + \ln(1+x^{2^n}) \\
&= \ln(1-x^2) + \ln(1+x^2) + \ln(1+x^4) + \cdots + \ln(1+x^{2^n}) \\
&= \ln(1-x^4) + \ln(1+x^4) + \cdots + \ln(1+x^{2^n}) \\
&= \ln(1-x^{2^n+1}) \rightarrow 0 \quad \text{as} \quad n \to \infty \quad \text{for} \quad |x| < 1.
\end{align*}
\]