#1 (Ch. 5: #15). Suppose that $f$ is a twice-differentiable real function on $(a, \infty)$, and $M_0, M_1, M_2$ are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$, respectively, on $(a, \infty)$. Prove that $M_0^2 \leq 4M_0M_2$.

#2 (Ch. 5: #17). Suppose $f$ is a real, three times differentiable function on $[-1, 1]$, such that
\[ f(-1) = 0, \quad f(0) = 0, \quad f(1) = 1, \quad f'(0) = 0. \]
Prove that $f^{(3)}(x) \geq 3$ for some point $x \in [-1, 1]$. Note that the above equalities hold for $f_0(x) := \frac{1}{2}(x^3 + x^2)$.

#3 (Ch. 5: #26). Suppose $f$ is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number $A$ such that $|f'(x)| \leq A |f(x)|$ on $[a, b]$. Prove that $f(x) \equiv 0$ on $[a, b]$.

#4. Let $f(x)$ be a real differentiable function on $[0, \infty)$ satisfying
\[ f'(x) = \cos(x^2) \cdot f(x) \quad \text{for all} \quad x \geq 0, \quad \text{and} \quad f(0) = 1. \]
Show that $e^{-x} \leq f(x) \leq e^x$ for all $x \geq 0$.

**Warning.** You cannot use integration!

#5. Let $f$ be a real continuous function on $\mathbb{R}$ such that $f(x) \equiv 0$ for $|x| \geq 1$. For fixed constant $\alpha \in (0, 1)$, introduce
\[
F_1 := \sup_{x \in \mathbb{R}, h > 0} \frac{|f(x + h) - f(x)|}{h^\alpha}, \quad F_2 := \sup_{x \in \mathbb{R}, h > 0} \frac{|f(x + h) - 2f(x) + f(x - h)|}{h^\alpha}.
\]
Show that if $F_1 < \infty$, then $F_1 \leq NF_2$, where $N$ is a constant depending only on $\alpha \in (0, 1)$.

**Hint.** Use the identity
\[ T - I = -\frac{1}{2}(T - I)^2 + \frac{1}{2}(T^2 - I), \quad \text{where} \quad Tf(x) := f(x + h). \]

#6. Define $f(x) := x \ln |x|$ for $0 < |x| \leq 1/2$, $f(0) := 0$. Show that
\[
F_1 := \sup_{|x| \leq 1/4, 0 < h < 1/4} \frac{|f(x + h) - f(x)|}{h} = \infty,
\]
\[
F_2 := \sup_{|x| \leq 1/4, 0 < h < 1/4} \frac{|f(x + h) - 2f(x) + f(x - h)|}{h} < \infty.
\]

**Remark.** One can extend $f(x)$ as a smooth function for $|x| \geq 1/2$ satisfying $f(x) \equiv 0$ for $|x| \geq 1$. Comparing with Problem #5, we see that the estimate $F_1 \leq NF_2$ fails in the case $\alpha = 1$. The a priori assumption $F_1 < \infty$ is actually not needed. Indeed, if $F_1^\varepsilon$ and $F_2^\varepsilon$ are defined in a similar way as $F_1$ and $F_2$, under an additional restriction $h \geq \varepsilon > 0$ with a small $\varepsilon > 0$, then by the same method we get $F_1^\varepsilon \leq NF_2^\varepsilon \leq NF_2$, and finally, $F_1 = \sup_{\varepsilon > 0} F_1^\varepsilon \leq NF_2$. 