Solutions of Homework #3:

Sec.2.3: #10; Sec.2.6: #2,4;
Sec.3.1: #2,10; Sec.3.2: #4,10; Sec.3.3: #4,6.

Grading is based on 5 problems not included in parentheses.

Sec.2.3: #6. Let

\( B_1 = \{ \text{the machine if properly adjusted} \} \), \( B_2 = \{ \text{the machine is not properly adjusted} \} \),

\( H = \{ \text{a selected part is of high quality} \} \), \( M = \{ \text{a selected part is of medium quality} \} \).

By assumptions,

\[ P(B_1) = \frac{9}{10}, \quad P(B_2) = \frac{1}{10}, \quad P(H|B_1) = \frac{1}{2}, \quad P(H|B_2) = \frac{1}{4}, \quad P(M|B_2) = \frac{3}{4}. \]

(a). Let \( A_1 \) that 4 out of 5 randomly selected items are of high quality and 1 item is of medium quality. There are \( \binom{5}{4} = 5 \) ways to select 4 items out of 5, and the probability of each selection is \( p^4q \), where \( q = 1 - p \), and \( p = P(H|B_1) = \frac{1}{2} \) in the case \( B_1 \), and \( p = P(H|B_2) = \frac{1}{4} \) in the case \( B_2 \). Hence

\[ P(A_1|B_1) = 5p^4q = 5 \cdot 2^{-5}, \quad P(A_1|B_2) = 5p^4q = 5 \cdot 3 \cdot 4^{-5}, \]

and by Bayes’ Theorem,

\[
P(B_1|A_1) = \frac{P(B_1) \cdot P(A_1|B_1)}{P(B_1) \cdot P(A_1|B_1) + P(B_2) \cdot P(A_1|B_2)} = \frac{9 \cdot 5 \cdot 2^{-5}}{9 \cdot 5 \cdot 2^{-5} + 0.1 \cdot 5 \cdot 3 \cdot 4^{-5}} = \frac{3 \cdot 2^5}{3 \cdot 2^5 + 1} = \frac{96}{97} \approx 0.98969.
\]

(b) Let \( M \) be the event that an additional item is of medium quality, and denote \( A_2 = A_1M \). By the Conditional Version of Bayes’ Theorem (2.3.4),

\[
P(B_1|A_2) = P(B_1|A_1M) = \frac{P(B_1|A_1) \cdot P(M|A_1B_1)}{P(B_1|A_1) \cdot P(M|A_1B_1) + P(B_2|A_1) \cdot P(M|A_1B_2)}.
\]

Here \( P(B_1|A_1) = \frac{96}{97} \), \( P(B_2|A_1) = 1 - P(B_1|A_1) = \frac{1}{97} \). Moreover, by conditions \( B_1 \) or \( B_2 \), the events \( A_1 \) and \( M \) are (conditionally) independent, so that

\[
P(M|A_1B_1) = P(M|B_1) = \frac{1}{2}, \quad P(M|A_1B_2) = P(M|B_2) = \frac{3}{4}.
\]
Therefore,
\[
P(B_1|A_2) = \frac{\frac{96}{97} \cdot \frac{2}{3}}{\frac{2}{3} \cdot 2 + 1} = \frac{32 \cdot 2}{32} = \frac{64}{65} \approx 0.98462.
\]

**An alternative way.** One can also calculate \( P(B_1|A_2) \) following the lines of (a). The event \( A_2 \) means 4 out of 6 randomly selected items are of high quality and 2 item is of medium quality. There are \( \binom{6}{4} = 15 \) ways to select 4 items out of 6, and the probability of each selection is \( p^4q^2 \). Now
\[
P(A_2|B_1) = \frac{15p^4q^2}{P(B_1)} = 15 \cdot 4^{-6} \cdot 9,
\]
\[
P(B_1|A_2) = \frac{P(B_1) \cdot P(A_2|B_1)}{P(B_1) \cdot P(A_2|B_1) + P(B_2) \cdot P(A_2|B_2)}
\]
\[
= \frac{0.9 \cdot 15 \cdot 2^{-6}}{0.9 \cdot 15 \cdot 2^{-6} + 0.1 \cdot 15 \cdot 4^{-6} \cdot 9} = \frac{2^6}{2^6 + 1} = \frac{64}{65} \approx 0.98462.
\]

**Sec.2.5: #2.** (a) The sample space consists of infinitely many events \( H_1H_2 \cdots H_{k-1}T_k \) and \( T_1T_2 \cdots T_{k-1}H_k \), where \( k = 1, 2, 3, \ldots \), \( H_k \) - the head on \( k \)-th tossing, \( T_k \) - the tail on \( k \)-th tossing. (b) Exactly three tosses are required in the cases \( H_1H_2T_3 \) and \( T_1T_2H_3 \). The probability of this event is
\[
P((H_1H_2T_3) \cup (T_1T_2H_3)) = P(H_1H_2T_3) + P(T_1T_2H_3) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.
\]

**Sec.2.5: #4.** For arbitrary \( A \) and \( B \), \( (A \cup B^c) B = (AB) \cup (B^cB) = (AB) \cup \emptyset = AB \). Since \( A \) and \( B \) are independent, we have \( P(AB) = P(A) \cdot P(B) \), hence
\[
P(A \cup B^c|B) = \frac{P((A \cup B^c) B)}{P(B)} = \frac{P(AB)}{P(B)} = P(A) = \frac{1}{3}.
\]

**Sec.3.1: #2.** From the equality
\[
1 = \sum_{x=1}^{5} f(x) = c \sum_{x=1}^{5} x = c \cdot (1 + 2 + 3 + 4 + 5) = c \cdot 15
\]

it follows \( c = \frac{1}{15} \).
Sec.3.1: #10 (a) By our assumptions the p.f. of $X;$

$$f(x) = c \cdot (x + 1) (8 - x) \quad \text{for} \quad x = 0, 1, \cdots, 7; \quad \text{and} \quad f(x) = 0 \quad \text{otherwise.}$$

From the equality

$$1 = \sum_{x=0}^{7} f(x) = c \cdot (8 + 14 + 18 + 20 + 20 + 18 + 14 + 8) = c \cdot 120$$

it follows $c = \frac{1}{120}$.

(b) We have

$$P(X \geq 5) = \sum_{x=5}^{7} f(x) = \frac{1}{120} \cdot (18 + 14 + 8) = \frac{40}{120} = \frac{1}{3}.$$

Sec.3.2: #4 (a). From the equality

$$1 = \int_{-\infty}^{\infty} f(x) \, dx = c \int_{1}^{2} x^2 \, dx = \frac{c}{3} [x^3]_{1}^{2} = \frac{7}{3} \cdot c$$

it follows $c = \frac{3}{7}$. (b) The probability

$$P\left(X > \frac{3}{2}\right) = \int_{3/2}^{\infty} f(x) \, dx = \frac{3}{7} \int_{3/2}^{2} x^2 \, dx = \frac{1}{7} [x^3]_{3/2}^{2} = \frac{1}{7} \cdot \left(8 - \frac{27}{8}\right) = \frac{37}{56}.$$

Sec.3.2: #10 (a). From the quality

$$1 = \int_{0}^{1} f(x) \, dx = c \int_{0}^{1} (1 - x)^{-1/2} \, dx = c \left[-2 (1 - x)^{1/2}\right]_{0}^{1} = 2c$$

it follows $c = \frac{1}{2}$.

(b) We have

$$P\left(X \leq \frac{1}{2}\right) = \int_{0}^{1/2} f(x) \, dx = \frac{1}{2} \left[-2 (1 - x)^{1/2}\right]_{0}^{1/2} = 1 - \frac{\sqrt{2}}{2} = 0.29289.$$

Sec.3.3: #6. The p.d.f. of $X,$

$$f(x) = \frac{dF(x)}{dx} = e^{x-3} \quad \text{for} \quad x \leq 3, \quad f(x) = 0 \quad \text{for} \quad x > 3.$$