Problem 1. Let $X$ be a random variable, which has a binomial distribution with parameters $n$ and $p$. It is known that $E(X) = 12$ and $Var(X) = 4$. Find $n$ and $p$.

Solution. For binomial distribution with parameters $n$ and $p$, we have $E(X) = np$ and $Var(X) = npq$. Therefore, we get a system

$$np = 12, \quad npq = 4,$$

which has a solution $q = \frac{1}{3}$, $p = \frac{2}{3}$, and $n = 18$.

Problem 2. Suppose that the $n$ random variables $X_1, \ldots, X_n$ form a random sample from the discrete distribution with

$$f(k) = \begin{cases} 2^{-(k+1)} & \text{for } k = 0, 1, 2, \ldots; \\ 0 & \text{otherwise.} \end{cases}$$

Find $P(X_1 = X_2 = \ldots = X_n)$.

Solution. We have

$$P(X_1 = X_2 = \ldots = X_n) = \sum_{k=0}^{\infty} P(X_1 = k) \cdot \ldots \cdot P(X_n = k)$$

$$= \sum_{k=0}^{\infty} [f(k)]^n = 2^{-n} \sum_{k=0}^{\infty} (2^{-n})^k = \frac{2^{-n}}{1 - 2^{-n}} = \frac{1}{2^n - 1}.$$
it follows $c = \frac{6}{7}$, and
\[
f_1(x) = \begin{cases} 
\frac{2}{7}(1 + 3x + 3x^2) & \text{for } 0 < x < 1, \\
0 & \text{otherwise.}
\end{cases}
\]
Since $f(x, y) \equiv f(y, x)$, the marginal p.d.f. of $Y$ is $f_2(y) \equiv f_1(y)$. Since $f(x, y) \neq f_1(x) \cdot f_2(y)$ for some $x$ and $y$, the random variables $X$ and $Y$ are not independent.

**Problem 4.** Suppose that two random variables $X$ and $Y$ have joint probability density function
\[
f(x, y) = \begin{cases} 
6x & \text{for } 0 < x < y < 1, \\
0 & \text{otherwise.}
\end{cases}
\]
Find the conditional probabilities:
(a) $P \left( X < \frac{1}{2} \mid Y < \frac{1}{2} \right)$, and
(b) $P \left( Y < \frac{1}{2} \mid X < \frac{1}{2} \right)$.

**Solution.** (a) Denote $A = \{ X < \frac{1}{2} \}$ and $B = \{ Y < \frac{1}{2} \}$. Since $f(x, y) \equiv 0$ for $x \geq y$, we have
\[
P(X \geq Y) = \int \int_{\{x \geq y\}} f(x, y) \, dx \, dy = 0.
\]
This means from $Y < \frac{1}{2}$ it follows $X < \frac{1}{2}$, i.e. $B \subset A$ and $AB = B$. Then
\[
\left( X < \frac{1}{2} \mid Y < \frac{1}{2} \right) = P(A \mid B) = \frac{P(AB)}{P(B)} = 1.
\]
(b) Further,
\[
P(A) = P \left( X < \frac{1}{2} \right) = \int \int_{\{x < \frac{1}{2}\}} f(x, y) \, dx \, dy = 6 \int_{0}^{\frac{1}{2}} \int_{x}^{1} y \, dx \, dy
\]
\[
= 6 \int_{0}^{\frac{1}{2}} (x - x^2) \, dx = \left[ \frac{3x^2 - 2x^3}{2} \right]_{0}^{\frac{1}{2}} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2},
\]
\[
P(B) = P \left( Y < \frac{1}{2} \right) = \int \int_{\{y < \frac{1}{2}\}} f(x, y) \, dx \, dy = 6 \int_{0}^{\frac{1}{2}} \int_{0}^{y} x \, dx \, dy = 3 \int_{0}^{\frac{1}{2}} y^2 \, dy = \frac{1}{8}.
\]
Therefore,
\[
P \left( Y < \frac{1}{2} \mid X < \frac{1}{2} \right) = P(B \mid A) = \frac{P(AB)}{P(A)} = \frac{P(B)}{P(A)} = \frac{1/8}{1/2} = \frac{1}{4}.
\]
Problem 5. Let $X_1$ and $X_2$ be independent with same distribution
\[ P(X = k) = qp^k \text{ for } k = 0, 1, 2, \ldots, \text{ where } 0 < p < 1, q = 1 - p. \]
Find the conditional probabilities
\[ P(X_1 = k \mid X_1 + X_2 = n) \text{ for } 0 \leq k \leq n. \]

Solution. Denote
\[ A_{n,j} = \{ X_1 = j, X_1 + X_2 = n \} = \{ X_1 = j, X_2 = n - j \}. \]
Since $X_1$ and $X_2$ are independent, we have
\[ P(A_{n,j}) = P(X_1 = j) \cdot P(X_2 = n - j) = qp^j \cdot qp^{n-j} = q^2 p^n \]
for all $j = 0, 1, \ldots, n$. Note that
\[ \{ X_1 + X_2 = n \} = \bigcup_{j=0}^{n} A_{n,j}, \]
where the events $A_{n,j}$ are disjoint. Therefore,
\[ P(X_1 + X_2 = n) = \sum_{j=0}^{n} P(A_{n,j}) = (n + 1)q^2 p^n. \]
Finally,
\[ P(X_1 = k \mid X_1 + X_2 = n) = \frac{P(X_1 = k, X_1 + X_2 = n)}{P(X_1 + X_2 = n)} = \frac{P(A_{n,k})}{P(X_1 + X_2 = n)} = \frac{1}{n + 1}. \]

Problem 6. Find the constants $c_1$ and $c_2$ such that the quantity
\[ W(X) = E(X^3) + c_1 E(X) \cdot E(X^2) + c_2 [E(X)]^3 \]
satisfies the property $W(X_1 + X_2) = W(X_1) + W(X_2)$ for any bounded and independent random variables $X_1$ and $X_2$.

Solution. One can either work with this expression directly, or use the equality
\[ \psi_{X_1+X_2}(t) = \psi_{X_1}(t) + \psi_{X_2}(t), \text{ where } \psi_X(t) = \ln \varphi_X(t), \varphi_X(t) = E(e^{tX}). \]
Then the desired property is satisfied for $W(X) = \psi''(0)$. We have
\[ \psi' = \frac{\varphi'}{\varphi}, \psi'' = \frac{\varphi'' \varphi - (\varphi')^2}{\varphi^2}, \psi''' = \frac{(\varphi''' \varphi + \varphi'' \varphi' - 2 \varphi' \varphi'') \varphi^2 - [\varphi'' \varphi - (\varphi')^2] \cdot 2 \varphi'}{\varphi^4}. \]
Since $\varphi(0) = 1$, we get
\[ \psi'''(0) = \varphi'''(0) - 3 \varphi'(0) \cdot \varphi''(0) + 2 [\varphi'(0)]^3 = E(X^3) - 3 E(X) \cdot E(X^2) + 2 [E(X)]^3, \]
so that $c_1 = -3$, $c_2 = 2$. 

3