Midterm 2, November 16, 2017. Problems and solutions

Problem 1. Suppose that random variables $X_1, X_2, X_3,$ and $X_4$ form a random sample from a uniform distribution on the interval $[0, 2]$, and let

$$Y = X_1 - 2X_2 + 3X_3 - 4X_4 + 5.$$ 

Find $E(Y)$ and $\text{Var}(Y)$.

Solution. For $i = 1, 2, 3, 4$, the p.d.f. of $X_i$ is

$$f(x) = \frac{1}{2} \text{ for } 0 \leq x \leq 2, \quad f(x) \equiv 0 \text{ otherwise.}$$

Hence $\mu = E(X_i) = \int_{-\infty}^{\infty} x f(x) \, dx = \frac{2}{0} \frac{1}{2} x \, dx = 1$, $E(X_i^2) = \int_{-\infty}^{\infty} x^2 f(x) \, dx = \frac{2}{0} \frac{1}{2} x^2 \, dx = \frac{4}{3}$, $\sigma^2 = \text{Var}(X_i) = E(X_i^2) - \mu^2 = \frac{1}{3}$, and

$$E(Y) = E(X_1) - 2E(X_2) + 3E(X_3) - 4E(X_4) + 5 = -2\mu + 5 = -\frac{8}{3} + 5 = -\frac{8}{3} + 5 = 3,$$

$$\text{Var}(Y) = \text{Var}(X_1) + (-2)^2 \text{Var}(X_2) + 3^2 \text{Var}(X_3) + (-4)^2 \text{Var}(X_4) = 30 \sigma^2 = 10.$$

Problem 2. Let $X$ be a random variable with distribution

$$P(X = k) = \frac{1}{2n+1} \text{ for } k = 0, \pm 1, \ldots, \pm n.$$ 

Evaluate $E(X)$ and $\text{Var}(X)$.

Solution. By symmetry, we have $\mu = E(X) = 0$,

$$\text{Var}(X) = E(X^2) - \mu^2 = \sum_{k=-n}^{n} k^2 \cdot P(X = k) = \frac{2}{2n+1} \sum_{k=1}^{n} k^2 = \frac{n(n + 1)}{3}.$$

Here we used the formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6}.$$

Problem 3. Suppose that the joint density of $X$ and $Y$ is

$$f(x, y) = \begin{cases} \frac{15}{4} x^2 & \text{for } 0 \leq y \leq 1 - x^2; \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal densities of $X$ and $Y$.

Solution. The marginal probability density functions (p.d.f.) of $X$ and $Y$,

$$f_1(x) = \int_{-\infty}^{\infty} f(x, y) \, dy = \int_{0}^{1-x^2} \frac{15}{4} x^2 \, dy = \frac{15}{4} x^2 (1 - x^2) \text{ for } -1 \leq x \leq 1,$$

$$= 0 \text{ otherwise;}$$

$$f_2(y) = \int_{-\infty}^{\infty} f(x, y) \, dx = \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{15}{4} x^2 \, dx = \frac{15}{2} \int_{0}^{\sqrt{1-y}} x^2 \, dx = \frac{5}{2} (1 - y)^{3/2} \text{ for } 0 \leq y \leq 1,$$

$$= 0 \text{ otherwise.}$$
**Problem 4.** Consider two random variables $X$ and $Y$, with joint probability density function
\[
f(x, y) = \begin{cases} 
4(x^3 + y^3) & \text{for } 0 < x < y < 1; \\
0 & \text{otherwise.}
\end{cases}
\]
Evaluate $E(X^2 + Y^2)$.

**Solution.**
\[
E(X^2 + Y^2) = \int\int (x^2 + y^2) f(x, y) \, dx \, dy = 4 \int_0^1 \int_y^1 (x^5 + x^3y^2 + x^2y^3 + y^5) \, dx \, dy
\]
\[
= 4 \int_0^1 \left( \frac{1}{6} + \frac{1}{4} + \frac{1}{3} + 1 \right) y^6 \, dy = 4 \cdot \frac{1}{7} \cdot \frac{1}{7} = 1.
\]

**Problem 5.** Let $X$ and $Y$ be independent random variables with same density
\[
f_1(x) = \begin{cases} 
e^{-x} & \text{for } x > 0, \\
0 & \text{otherwise;}
\end{cases}
f_2(y) = \begin{cases} 
e^{-y} & \text{for } y > 0, \\
0 & \text{otherwise.}
\end{cases}
\]
Find the distribution function of the random variable $Z = \frac{X}{X+Y}$.

**Solution.** Since $X$ and $Y$ are independent, their joint density
\[
f(x, y) = \begin{cases} 
e^{-x-y} & \text{for } x, y > 0; \\
0 & \text{otherwise.}
\end{cases}
\]
Obviously, $0 < Z = \frac{X}{X+Y} < 1$ with probability 1. For $0 < z < 1$, we have
\[
F(z) = P\left(Z = \frac{X}{X+Y} < z\right) = \int\int e^{-x-y} \, dx \, dy
\]
where
\[
D = \left\{(x, y): x, y > 0, \frac{x}{x+y} < z\right\} = \left\{(x, y): 0 < x < \infty, \frac{x}{z} > y > \frac{x(1-z)}{z}\right\}.
\]
We can write
\[
F(z) = \int_0^\infty e^{-x} \left[ \int_{\frac{x}{z}}^\infty e^{-y} \, dy \right] \, dx = \int_0^\infty e^{-x} e^{\frac{x(1-z)}{z}} \, dx = \int_0^\infty e^{-x} \, dx = z.
\]
This means that $Z$ has uniform distribution on the interval $[0, 1]$.

**Problem 6.** Let $X$ be a random variable such that
\[
P(X = n) = \frac{1}{2^n+1} \quad \text{for } n = 0, 1, 2, \ldots.
\]
Find the moment generating function $\psi$ for $X$, and then use it to find $E(X)$.

**Solution.**
\[
\psi(t) = E e^{tx} = \sum_{n=0}^{\infty} \frac{e^{tn}}{2^n+1} = \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{e^t}{2} \right)^n = \frac{1}{2} \cdot \frac{1}{1 - \frac{e^t}{2}} = \frac{1}{2 - e^t} \quad \text{for small } |t|.
\]
Therefore,
\[
\psi'(t) = \frac{e^t}{(2 - e^t)^2}, \quad \text{and } E(X) = \psi'(0) = 1.
\]