Problem 1. Let $A, B, C$ be three arbitrary events. Find expressions for the event that of $A, B, C$ (just write the answers without explanations):

(a) Only $A$ occurs. \[ A B^c C^c \]

(b) Both $A$ and $B$, but not $C$, occur. \[ A B C^c \]

(c) All three events occur. \[ A B C \]

(d) At least one occurs. \[ A \cup B \cup C \]

(e) At least two occur. \[ (A B) \cup (A C) \cup (B C) \]

(f) One and no more occurs. \[ (A B^c C^c) \cup (A^c B C^c) \cup (A^c B^c C) \]

(g) Two and no more occur. \[ (A^c B C) \cup (A B^c C) \cup (A B C^c) = \]

\[ = \text{exactly two} \]

\[ = (A \cap B) \cup (A \cap C) \cup (B \cap C) \]

(h) None occurs. \[ A^c B^c C^c \]

(i) Not more than two occur. \[ (A B C)^c = A^c \cup B^c \cup C^c \]

\[ = \text{not three} \]
Problem 2. Two dice are thrown. Let $A$ be the event that the sum of the faces is odd, $B$ the event of at least one "1". Find the probabilities of the events $AB$, $A \cup B$, $AB^c$ assuming that all 36 sample points have same probability.

$$A = \{\text{sum of faces is odd}\}.$$  
Since the outcomes are equally split between "even+odd" and "odd+even", we have $P(A) = \frac{1}{2}$.

$$B = \{\text{at least one "1"}\} = B_1 \cup B_2,$$
where $B_k = \{k^{th} \text{ die shows "1"}\}$, $k=1,2$.

Since $B_1$ and $B_2$ are independent with $P(B_k) = \frac{1}{6}$, we have

$$P(B) = P(B_1) + P(B_2) - P(B_1 B_2) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}.$$  
Further,

$$AB_1 = \{1^{st} \text{ die is "1", 2nd die is even}\},$$  
$$AB_2 = \{2^{nd} \text{ die is "1", 1st die is even}\}.$$  
These two events are disjoint with $P(AB_1) = P(AB_2) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$. Then

$$P(AB) = P(AB_1) + P(AB_2) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}.$$  

$$P(A \cup B) = P(A) + P(B) - P(AB) = \frac{1}{2} + \frac{11}{36} - \frac{1}{6} = \frac{23}{36}.$$  

$$P(AB^c) = P(A) - P(AB) = \frac{1}{2} - \frac{1}{12} = \frac{5}{12}.$$
Problem 3. A certain unfair coin gives a head with probability 1/3 when tossed. As an experiment, a player tosses the coin five times.

(a) Find the probability that at least one head is obtained in the five tosses.
(b) Given that at least one head was obtained in the five tosses, find the probability that a head was obtained on both the third and fourth tosses.

Let $H_j$ be the event that a head was obtained on toss $j$ for $j = 1, 2, 3, 4, 5$.

(a) $P(A) = P(\text{at least one } H_j) = P \left( \bigcup_{j} H_j \right) = 1 - P(H_1^c H_2^c H_3^c H_4^c H_5^c) = 1 - \left( \frac{2}{3} \right)^5 = \frac{211}{243} \approx 0.868$

(b) $P(H_3 H_4 | A) = \frac{P(H_3 H_4)}{P(A)} = \frac{\left( \frac{1}{3} \right)^2}{\frac{211}{243}} = \frac{27}{211} \approx 0.128$
Problem 4. Prove that if $A$, $B$, and $C$ are independent, then $A$ and $B \cup C$ are independent.

\[
P(A \ (B \cup C)) = P((AB) \cup (AC)) = \\
= P(AB) + P(AC) - P(ABC) \\
= P(A) \cdot \left[ P(B) + P(C) - P(B) \cdot P(C) \right] \\
= P(A) \cdot P(B \cup C),
\]

i.e. $A$ and $B \cup C$ are independent.
Problem 5. Suppose that $A$ and $B$ are two events such that

$$P(A|B) = 0.7, \quad P(A|B^c) = 0.3, \quad P(B|A) = 0.6.$$ 

Find $P(A)$.

Denote $\alpha = P(A)$. Then

$$P(AB) = P(A) \cdot P(B|A) = \frac{3}{5} \cdot \alpha.$$ 

$$P(B) = \frac{P(AB)}{P(A|B)} = \frac{10}{7} \cdot \frac{3}{5} \cdot \alpha = \frac{6}{7} \cdot \alpha.$$ 

By Law of Total Probability (Theorem 2.1.4),

$$\alpha = P(A) = P(B) \cdot P(A|B) + P(B^c) \cdot P(A|B^c)$$

$$= \frac{7}{10} \cdot \frac{6}{7} \cdot \alpha + \frac{3}{10} \cdot \left(1 - \frac{6}{7} \cdot \alpha\right) = \frac{3}{10} + \frac{12}{35} \cdot \alpha,$$

$$\frac{23}{35} \cdot \alpha = \frac{3}{10}, \quad \alpha = \frac{3}{10}, \quad \frac{35}{23} = \frac{21}{46} \approx 0.4565.$$
Problem 6. Let $X$ and $Y$ be random variables with probability functions (p.f.)

$$P(X = k) = \frac{1}{8} \binom{3}{k} \quad \text{for} \quad k = 0, 1, 2, 3;$$

$$P(Y = j) = \frac{1}{2} \quad \text{for} \quad j = 0, 1.$$

Moreover, let events $\{X = k\}$ and $\{Y = j\}$ be independent for each pair $k, j$. Find the p.f. of the random variable $Z = X + Y$.

One can think of $X$ and $Y$ as the number of heads (successes) in a sequence of 3 and 1 independent tossings of a fair coin. Then $Z = X + Y$ is the number of heads in a sequence of 4 tossings of a fair coin, which has binomial distribution with parameters $n = 4$ and $p = \frac{1}{2}$.

$$p_k = P(Z = k) = \binom{4}{k} \left( \frac{1}{2} \right)^4 = \frac{1}{16} \binom{4}{k} \quad \text{for} \quad k = 0, 1, 2, 3, 4.$$

Namely,

$$p_0 = \frac{1}{16}, \quad p_1 = \frac{4}{16} = \frac{1}{4}, \quad p_2 = \frac{6}{16} = \frac{3}{8}, \quad p_3 = p_1 = \frac{1}{4}, \quad p_4 = p_0 = \frac{1}{16}. $$