Solutions of Homework #5:

Sec.3.8: #2, 8; Sec.3.9: #4, 6; Sec.3.11: #20;
Sec.4.1: #8; Sec.4.2: #2, 8, (10).

Grading is based on 5 problems not included in parentheses.

Sec.3.8: #2. We have \( Y = r(X) \), where \( r(x) = x^2 - x = x(x - 1) \), so that
\[
\begin{align*}
  r(-3) &= 12, & r(-2) &= r(3) = 6, & r(-1) &= r(2) = 2, & r(0) &= r(1) = 0.
\end{align*}
\]
Therefore, the p.f. of \( Y \),
\[
g(y) = P(Y = y) = \sum_{x: r(x) = y} \frac{1}{7}
\]
is determined by the equalities
\[
\begin{align*}
g(12) &= \frac{1}{7}, & g(6) &= g(2) = g(0) = \frac{2}{7}.
\end{align*}
\]

Sec.3.8: #8. We have \( Y = r(X) \), where \( r(x) = \sqrt{x} \) is a monotone function on \([0, \infty)\). The inverse function \( x = s(y) = y^2 \) on \([0, \infty)\). By the formula (3.8.3), the p.d.f. of \( Y \) is
\[
g(y) = \begin{cases} 
  f(s(y)) \cdot \left| \frac{ds(y)}{dy} \right| = e^{-y^2} \cdot 2y & \text{for } y \geq 0, \\
  0 & \text{otherwise}.
\end{cases}
\]

Sec.3.9: #4. Define \( Y = r(X) \) by the equalities \( Y_1 = X_1 X_2, \; Y_2 = X_2 \). Notice that the image of the square \( S = \{0 < x_1 < 1, \; 0 < x_2 < 1\} \) by the transformation \( y = r(x) \) is the triangle \( T = \{0 < y_1 < y_2 < 1\} \). The inverse transformation \( x = s(y) \) is defined by formulas \( x_1 = y_1 y_2^{-1}, \; x_2 = y_2 \), so that the Jacobian
\[
J = \det \left( \frac{\partial x}{\partial y} \right) = \det \begin{pmatrix} y_2^{-1} & -y_1 y_2^{-2} \\ 0 & 1 \end{pmatrix} = y_2^{-1}.
\]
By the formula (3.9.13), the joint p.d.f. of \( Y_1 \) and \( Y_2 \) is
\[
g(y_1, y_2) = \begin{cases}
  f(x_1, x_2) \cdot |J| = (y_1 y_2^{-1} + y_2) \cdot y_2^{-1} = y_1 y_2^{-2} + 1 & \text{for } 0 < y_1 < y_2 < 1, \\
  0 & \text{otherwise}.
\end{cases}
\]
Finally, the p.d.f. of \( Y_1 = X_1 X_2 \) is
\[
g_1(y_1) = \int_{y_1}^\infty g(y_1, y_2) dy_2 = \begin{cases}
  \frac{1}{y_1} \int \left( y_1 y_2^{-2} + 1 \right) dy_2 = 2 (1 - y_1) & \text{for } 0 < y_1 < 1, \\
  0 & \text{otherwise}.
\end{cases}
\]
Sec.3.9: #6. Define $Z = (Z_1, Z_2) = r(X, Y)$ by the equalities $Z_1 = X + Y$, $Z_2 = Y$. The inverse transformation $(x, y) = s(z_1, z_2)$ is defined by formulas $x = z_1 - z_2$, $y = z_2$, so that the Jacobian

$$J = \det \left( \frac{\partial (x, y)}{\partial (z_1, z_2)} \right) = \det \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) = 1.$$ 

The image of the triangle $S = \{0 \leq x \leq y \leq 1\}$ by the transformation $z = r(x, y)$ is the triangle $T = \{0 \leq z_1 - z_2 \leq z_2 \leq 1\} = \{0 \leq z_2 \leq 1, \quad z_2 \leq z_1 \leq 2z_2\}$

with vertices $(0, 0)$, $(1, 1)$, and $(2, 1)$. By the formula (3.9.13) in the textbook, the joint p.d.f. of $Z_1$ and $Z_2$ is

$$g(z_1, z_2) = \begin{cases} f(x_1, x_2) \cdot |J| = 2z_1 & \text{for } z = (z_1, z_2) \in T, \\ 0 & \text{otherwise.} \end{cases}$$

The section of $T$ for fixed $z_1 \in (0, 2)$ is a segment of length

$$l(z_1) = \begin{cases} z_1/2 & \text{for } 0 \leq z_1 \leq 1, \\ (2 - z_1)/2 & \text{for } 1 < z_1 \leq 2. \end{cases}$$

Therefore, the p.d.f. of $Z_1 = X + Y$ is

$$g_1(z_1) = \int_{-\infty}^{\infty} g(z_1, z_2) \, dz_2 = 2z_1 l(z_1) = \begin{cases} z_1^2 & \text{for } 0 \leq z_1 \leq 1, \\ z_1(2 - z_1) & \text{for } 1 < z_1 \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Sec.3.11: #20. Note that the joint p.d.f. $f(x, y, z)$ of $X, Y, Z$ can be factored in the product of $f_1(x, y)$ - the joint p.d.f. of $X, Y$, and $f_2(z)$ - the p.d.f. of $Z$, where

$$f_1(x, y) = \begin{cases} 2 & \text{for } 0 < x < y < 1, \\ 0 & \text{otherwise;} \end{cases} \quad f_2(z) = \begin{cases} 1 & \text{for } 0 < z < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $(X, Y)$ and $Z$ are independent, so that the events $A = \{3X > Y\}$ and $B = \{1 < 4Z < 2\}$ are independent. Therefore,

$$P(3X > Y \mid 1 < 4Z < 2) = P(A \mid B) = P(A) = \frac{1}{2} \int_{0}^{1} \left( \int_{y/3}^{y} 2 \, dx \right) \, dy = \int_{0}^{1} \frac{2}{3} y \, dy = \frac{2}{3}.$$ 

Sec.4.1: #8. We have

$$E(XY) = \int_{0}^{1} \int_{0}^{x} xyf(x, y) \, dx \, dy = \int_{0}^{1} x \int_{0}^{x} 12y^3 \, dy \, dx = \int_{0}^{1} 3x^5 \, dx = \frac{1}{2} = 0.5.$$
Sec.4.2: #2. Since \( E(X_1) = E(X_2) = E(X_3) = 5 \), we have

\[
E(2X_1 - 3X_2 + X_3 - 4) = 2E(X_1) - 3E(X_2) + E(X_3) - 4 = -4.
\]

Sec.4.2: #8. Each of girls or boys can be selected with probability \( p = \frac{8}{25} \). We have

\[
X = X_1 + \cdots + X_{10}, \quad \text{where} \quad X_i = 1 \quad \text{if} \quad i\text{-th boy was selected, and} \quad X_i = 0 \quad \text{otherwise};
\]

\[
Y = Y_1 + \cdots + Y_{15}, \quad \text{where} \quad Y_i = 1 \quad \text{if} \quad i\text{-th girl was selected, and} \quad Y_i = 0 \quad \text{otherwise}.
\]

Since \( E(X_i) = E(Y_i) = 0 \cdot (1 - p) + 1 \cdot p = p \) for each \( i \),

\[
E(X - Y) = \sum_{i=1}^{10} E(X_i) - \sum_{i=1}^{15} E(Y_i) = 10p - 15p = -5p = -\frac{8}{5} = -1.6.
\]

Sec.4.2: #10(a). This example has been considered in class in a more general case when \( P(\text{Head}) = p, \ 0 < p < 1 \). Let \( X \) denote the number of tosses until a head is obtained.

\[
P(X = k) = P(T_1T_2\cdots T_{k-1}H_k) = p^{k-1}q \quad \text{for} \quad k = 1, 2, \cdots \quad (1)
\]

Therefore,

\[
E(X) = \sum_{k=1}^{\infty} kP(X = k) = \sum_{k=1}^{\infty} kq^{k-1}p = p \sum_{k=1}^{\infty} \frac{d}{dq} (q^k) = p \frac{d}{dq} \sum_{k=0}^{\infty} q^k
\]

\[
= p \frac{d}{dq} \left[ (1 - q)^{-1} \right] = p (1 - q)^{-2} = \frac{1}{p} = 2.
\]

(b). The number of tails before the first head is obtained is \( Y = X - 1 \), so that

\[
E(Y) = E(X) - 1 = 1.
\]