Math 5652: Introduction to Stochastic Processes: Fall 2014

Appendix A. Generating functions

Let $X$ be a random variable with values in $\mathbb{R}^1$. The generating, or probability generating function of $X$ is defined as $\phi(t) = \phi_X(t) = E(t^X)$. If $X$ has discrete distribution with $p_k = P(X = k)$, $k = 0, 1, 2, \ldots$, then

$$\phi(t) = \sum_{k=0}^{\infty} p_k t^k \text{ for } |t| \leq 1, \text{ and } p_k = \frac{\phi^{(k)}(0)}{k!}.$$  

In this case, we also have

$$\phi(1) = 1, \quad \phi^{(k)}(1) = E(X(X - 1) \cdots (X - k + 1)) \text{ for } k = 1, 2, \ldots.$$  

The moment generating function of $X$ is $\varphi(t) = \varphi_X(t) = E(e^{tX}) = \phi(e^t)$. If it is defined in a neighborhood of the point $t = 0$, then the $k^{th}$ moment of $X$,

$$E(X^k) = \varphi^{(k)}(0), \quad k = 1, 2, \ldots.$$  

In this case, we also have

$$E(X) = \psi'(0), \quad \text{Var}(X) = \psi''(0), \quad \text{where } \psi(t) = \ln \varphi(t).$$  

Note that if $X_1, X_2, \ldots, X_n$ are independent, and $X = X_1 + X_2 + \cdots + X_n$, then

$$\varphi_X = \varphi_{X_1} \cdot \varphi_{X_2} \cdots \varphi_{X_n}, \quad \psi_X = \psi_{X_1} + \psi_{X_2} + \cdots + \psi_{X_n}.$$

1. $X = \text{Binomial}(n, p)$ with $n = 1, 2, \ldots; 0 \leq p \leq 1$.

   $$f_1(k) = P(X = k) = \binom{n}{k} p^k q^{n-k} \text{ for } k = 0, 1, \ldots, n; \quad \text{where } q = 1 - p; \quad \phi_1(t) = E(t^X) = (pt + q)^n, \quad \varphi_1(t) = E(e^{tX}) = (pe^t + q)^n, \quad \mu_1 = E(X) = np, \quad \sigma_1^2 = \text{Var}(X) = npq.$$

2. $X = \text{Poisson}(\lambda)$ with $\lambda > 0$.

   $$f_2(k) = P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!} \text{ for } k = 0, 1, 2 \ldots; \quad \phi_2(t) = E(t^X) = \exp(\lambda(t - 1)), \quad \varphi_2(t) = E(e^{tX}) = \exp(\lambda(e^t - 1)), \quad \mu_2 = E(X) = \lambda, \quad \sigma_2^2 = \text{Var}(X) = \lambda.$$

3. $X = \text{Negative Binomial}(r, p)$ - the number of trials with probability of success $p$ until $r^{th}$ success.

   $Y = X - r = \text{Shifted Negative Binomial}(r, p)$ - the number of failures before $r^{th}$ success. We have

   $$P(X = k) = \binom{k-1}{r-1} p^r q^{k-r} \text{ for } k = r, r + 1, \ldots; \quad P(Y = j) = P(X = r + j) = \binom{r + j - 1}{j} p^r q^j = \binom{r + j - 1}{j} p^r q^j \text{ for } j = 0, 1, 2, \ldots.$$  

   Note that by Taylor’s formula, for $a \in \mathbb{R}^1$ and $|t| < 1$,

   $$g(t) = (1 + t)^a = 1 + \sum_{k=1}^{\infty} \frac{g^{(k)}(0)}{k!} t^k = 1 + \sum_{k=1}^{\infty} \frac{a}{k} t^k, \quad \text{where } \binom{a}{k} = \frac{a(a-1) \cdots (a-k+1)}{k!}.$$  

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If $a = n$ is a natural number, then this equality is reduced to the binomial formula (for all $t \in \mathbb{R}^1$):
\[ g(t) = (1 + t)^n = 1 + \sum_{k=1}^{n} \binom{n}{k} t^k, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}. \]

Substituting $t = -q$ and $a = -r$, we get
\[ 1 = p^r (1 - q)^{-r} = p^r \cdot \left(1 + \frac{r}{1!} q + \frac{r(r+1)}{2!} q^2 + \cdots \right) = \sum_{j=0}^{\infty} P(Y = j). \]

Note that the above distributions are well defined for all $r \in \mathbb{R}^1$, correspondingly,
\[ \phi(t) = E(t^Y) = \sum_{j=0}^{\infty} t^j P(Y = j) = \sum_{j=0}^{\infty} \binom{r + j - 1}{j} p^r (tq)^j = \left(\frac{p}{1 - qt}\right)^r; \]
\[ \varphi_Y(t) = E(e^{tY}) = \left(\frac{p}{1 - qt}\right)^r, \quad \varphi_X(t) = E(e^{tX}) = e^{tr} E(e^{tY}) = \left(\frac{p}{e^{-t} - q}\right)^r; \]
\[ \mu_Y = E(Y) = \frac{rq}{p}, \quad \mu_X = E(X) = \mu_Y + r = \frac{r}{p}, \quad \sigma_X^2 = \text{Var}(X) = \text{Var}(Y) = \frac{rq}{p^2}. \]

3a. Geometric $(p)$ = Negative Binomial $(1, p)$.

4. $X = \text{Gamma} (\alpha, \lambda)$, where $\alpha > 0, \lambda > 0$, if it has density
\[ f_4(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \quad \text{for} \quad x > 0, \quad \text{and} \quad f_4(x) = 0 \quad \text{otherwise}. \]

Here $\Gamma(\alpha)$ denotes the Gamma function:
\[ \Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} \, dx, \]
which satisfies the properties
\[ \Gamma(\alpha + 1) = \alpha \Gamma(\alpha), \quad \Gamma(n + 1) = n!, \quad \Gamma(1/2) = \sqrt{\pi}. \]

We have
\[ \varphi_4(t) = E(e^{tX}) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha, \quad \mu_4 = E(X) = \frac{\alpha}{\lambda}, \quad \sigma_4^2 = \text{Var}(X) = \frac{\alpha}{\lambda^2}. \]

4a. Exponential $(\lambda)$ = Gamma $(1, \lambda)$.

5. $X = \text{Normal} (\mu, \sigma^2)$ is related to $Y = \text{Standard Normal} = \text{Normal} (0, 1)$ by the formula $X = \mu + \sigma \cdot Y$. The corresponding densities
\[ f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \quad f_X(x) = \frac{1}{\sigma} f_Y\left(\frac{x - \mu}{\sigma}\right) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(x - \mu)^2}{2\sigma^2} \right]. \]

We have
\[ \varphi_Y(t) = E(e^{tY}) = e^{t^2/2}, \quad \varphi_X(t) = E(e^{tX}) = \exp \left( t\mu + \frac{t^2\sigma^2}{2} \right); \]
\[ \mu_Y = E(Y) = 0, \quad \sigma_Y^2 = \text{Var}(Y) = 1; \quad \mu_X = E(X) = \mu, \quad \sigma_X^2 = \text{Var}(X) = \sigma^2. \]