Laplace’s Equation

Let $\Omega$ be an open set in $\mathbb{R}^n$. A function $u \in C^2(\Omega)$ is called harmonic in $\Omega$ if it satisfies Laplace’s equation

$$\Delta u := \sum_{i=1}^{n} D_{ii} u = 0 \quad \text{in} \quad \Omega.$$  

A function $u \in C^2(\Omega)$ is called subharmonic (superharmonic) in $\Omega$ if $\Delta u \geq 0$ ($\Delta u \leq 0$) in $\Omega$. In this chapter, we discuss some basic properties of harmonic functions, such as mean value theorems, maximum principles, Harnack inequalities, etc. Note that Laplace’s equation (1.1) is a special case of Poisson’s equation

$$\Delta u = f \quad \text{in} \quad \Omega.$$  

1.1. Mean Value Formulas

Let $u \in C^2(\Omega)$. From the divergence theorem it follows

$$\int_{\Omega_0} \Delta u \, dx = \int_{\Omega_0} \text{div} (Du) \, dx = \int_{\partial \Omega_0} Du \cdot \nu \, dS =: \int_{\partial \Omega_0} \frac{\partial u}{\partial \nu} \, dS$$

for any bounded open subset $\Omega_0 \subset \overline{\Omega_0} \subset \Omega$ with smooth boundary $\partial \Omega_0$. Here $dS$ is the $(n-1)$-dimensional surface area element in $\partial \Omega_0$, and $\nu$ is the unit outward normal to $\partial \Omega_0$.

Fix $y \in \Omega$ and $r > 0$, such that $B_r := B_r(y) \subset \overline{B}_r \subset \Omega$. For $0 < \rho \leq r$, consider the integral mean value of $u$ over the surface $\partial B_\rho(y)$:

$$m(\rho) := \frac{1}{|\partial B_\rho|} \int_{\partial B_\rho(y)} u(x) \, dS_x = \frac{1}{\sigma_n} \int_{\partial B_{\rho}(0)} u(y + \rho z) \, dS_z,$$
where \(|\partial B_\rho| = \rho^{n-1} |\partial B_1| = \rho^{n-1} \sigma_n\) – the \((n-1)\)-dimensional surface area of \(\partial B_\rho\). Since \(u(x)\) is continuous, \(m(\rho) \to u(y)\) as \(\rho \to 0^+\). Further, \(z = (x - y)/\rho =: \nu_x\) – the unit outward normal to \(\partial B_\rho(y)\) at the point \(x = y + \rho z\). Applying the identity (1.3) to the ball \(B_\rho(y)\), we have

\[
m'(\rho) = \frac{1}{\sigma_n} \int_{\partial B_1(0)} Du(y + \rho z) \cdot z dS_z
\]

\[
= \frac{\rho^{1-n}}{\sigma_n} \int_{\partial B_\rho(y)} Du(x) \cdot \nu_x dS_x = \frac{\rho^{1-n}}{\sigma_n} \int_{B_\rho(y)} \Delta u(x) dx,
\]

and

\[
m(\rho) - u(y) = \frac{1}{\sigma_n} \int_0^\rho m'(\rho') d\rho' = \frac{1}{\sigma_n} \int_0^\rho \rho^{1-n} \Delta u(x) dx d\rho
\]

(1.5)

As a byproduct, here we get the following mean value properties.

**Theorem 1.1** (Mean value formulas). Let \(u\) be a function in \(C^2(\Omega)\) satisfying \(\Delta u = (\geq, \leq) = 0\) in \(\Omega\). Then for any ball \(B_r(y) \subset B_r(y) \subset \Omega\), we have

\[
(1.6) \quad u(y) = (\leq, \geq) m(r) := \frac{1}{|\partial B_r|} \int_{\partial B_r(y)} u(x) dS_x,
\]

\[
(1.7) \quad u(y) = (\leq, \geq) \frac{1}{|B_r|} \int_{B_r(y)} u(x) dx.
\]

**Proof.** The relation (1.6) follows immediately from (1.4)–(1.5). In order to get (1.7), one can integrate the relation

\[
\int_{\partial B_\rho(y)} u(y) \cdot |\partial B_\rho| = (\leq, \geq) \int_{\partial B_\rho(y)} u(x) dS_x
\]

with respect to \(\rho\) from 0 to \(r\). \(\Box\)
1.2. The Maximum Principle

From the mean value formula (1.7), it is easy to derive the following strong maximum principle for subharmonic functions.

**Theorem 1.2** (Strong maximum principle). Let \( \Omega \) be an open connected set in \( \mathbb{R}^n \), and let \( u \in C^2(\Omega) \) satisfy \( \Delta u \geq 0 \) in \( \Omega \). Suppose there is a point \( x_0 \in \Omega \) such that \( u(x_0) = \sup_{\Omega} u \). Then \( u = \text{const} \) in \( \Omega \).

**Proof.** Denote \( M := \sup_{\Omega} u \), and \( \Omega' := \{ x \in \Omega : u(x) = M \} \).

The set \( \Omega' \) is not empty, because it contains the point \( x_0 \). For an arbitrary point \( y \in \Omega' \), one can choose \( r > 0 \) such that \( B_r(y) \subset B_r(y) \subset \Omega \). By virtue of (1.7), we have

\[
M = u(y) \leq \frac{1}{|B_r|} \int_{B_r(y)} u(x) \, dx \leq M,
\]

hence \( u \equiv M \) in \( B_r(y) \), i.e. \( B_r(y) \subset \Omega' \). Therefore, the set \( \Omega' \) is both open and closed relative to \( \Omega \). Since \( \Omega \) is connected, we must have \( \Omega' = \Omega \). \( \square \)

In turn, from the strong maximum principle it follows immediately the following weak maximum principle, which is usually referred to as the maximum principle.

**Theorem 1.3** (Weak maximum principle). Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \), and let \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) satisfy \( \Delta u \geq 0 \) in \( \Omega \). Then

\[
(1.8) \quad \sup_{\Omega} u = \sup_{\partial \Omega} u.
\]

**Remark 1.4.** The minimum principles for superharmonic functions are formulated similarly to Theorems 1.2 and 1.3, one only needs to replace the assumption \( \Delta u \geq 0 \) with \( \Delta u \leq 0 \), and the sup with the inf. The proof follows from the above theorems applied to the function \(-u\).

As easy consequences of the maximum principle, we obtain the following two theorems.

**Theorem 1.5** (Comparison principle). Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \), and let functions \( u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega}) \) satisfy \( \Delta u_1 \geq \Delta u_2 \) in \( \Omega \), and \( u_1 \leq u_2 \) on \( \partial \Omega \). Then \( u_1 \leq u_2 \) in \( \Omega \).

**Proof.** This assertion follows from Theorem 1.3 applied to the function \( u := u_1 - u_2 \). \( \square \)
Theorem 1.6 (Uniqueness). Let \( f \in C(\Omega) \) and \( f \in C(\partial \Omega) \) be given functions, where \( \Omega \) is a bounded open set in \( \mathbb{R}^n \). Then there exists at most one classical solution, i.e., solution \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) of the boundary value problem

\[
\Delta u = f \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial \Omega.
\]

**Proof.** If \( u_1 \) and \( u_2 \) both satisfy (1.9), then Theorem 1.3 applied to \( u = \pm(u_1 - u_2) \) guarantees \( u_1 \equiv u_2 \) in \( \Omega \).

### 1.3. The Harnack Inequality

The following Harnack inequality is another consequence of the mean value formula (1.7) for harmonic functions.

**Theorem 1.7** (Harnack inequality). Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \), and a constant \( \delta > 0 \) be such that the set \( \Omega^{\delta} := \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \delta \} \) is connected. Then for any non-negative harmonic function \( u \) in \( \Omega \),

\[
\sup_{\Omega^{\delta}} u \leq N \inf_{\Omega^{\delta}} u,
\]

where the constant \( N \) depends only on \( n \) and \( \delta / \text{diam} \Omega \).

**Proof.** Let \( r := \delta / 2 \). Then for any two points \( x_0, x_1 \) in \( \Omega^{\delta} \), such that \( |x_0 - x_1| \leq r \), we have \( B_r(x_0) \subset B_{2r}(x_1) \subset \Omega \). Since \( u \geq 0 \) in \( \Omega \), this implies

\[
\int_{B_r(x_0)} u(x) \, dx \leq \int_{B_{2r}(x_1)} u(x) \, dx.
\]

In turn, by the equality (1.7), this yields \( |B_r| \cdot u(x_0) \leq |B_{2r}| \cdot u(x_1) \), so that \( u(x_0) \leq 2^n u(x_1) \).

Note that for any \( x, y \in \Omega^{\delta} \), one can choose a finite sequence \( x = x_0, x_1, \ldots, x_m = y \) in \( \Omega^{\delta} \), such that \( |x_j - x_{j-1}| \leq r \) for all \( j = 1, \ldots, m \), and \( m \) does not exceed a constant \( M = M(n, \delta / \text{diam} \Omega) \). By the previous argument,

\[
u(x) = u(x_0) \leq 2^n u(x_1) \leq 2^{2n} u(x_2) \leq \ldots \leq 2^{mn} u(x_m) = 2^{mn} u(y) \leq 2^{Mn} u(y),
\]

and the desired estimate (1.10) follows with \( N := 2^{Mn} \).

**Corollary 1.8** (Liouville theorem). Any harmonic function defined over \( \mathbb{R}^n \) and bounded below (or above) is constant.
1.4. Regularity of harmonic functions

Proof. Replacing $u$ by $\pm u + \text{const}$, we may assume that $\inf u = 0$. Applying the Harnack inequality with $\Omega = B_{2r}(0)$, $\delta = r$, we obtain

$$0 \leq \sup_{B_r(0)} u \leq N \inf_{B_r(0)} u \to 0 \quad \text{as} \quad r \to \infty.$$  

Since the left side is a non-decreasing function of $r$, we get $u \equiv 0$. \hfill \Box

1.4. Regularity of harmonic functions

In this section, we show that the mean value formula implies the interior smoothness, and even analyticity, of harmonic functions. We will use so-called mollifiers $\eta^\delta(x)$, $\delta = \text{const} > 0$, satisfying the properties

$$\eta^\delta \in C^\infty(\mathbb{R}^n), \quad \eta^\delta \equiv 0 \quad \text{on} \quad \mathbb{R}^n \setminus B_\delta(0), \quad \text{and} \quad \int_{\mathbb{R}^n} \eta^\delta(x) \, dx = 1. \quad (1.11)$$

We also assume that $\eta^\delta$ are radially symmetric, i.e. $\eta^\delta(x) = \eta^\delta(|x|)$, where the functions $\eta^\delta_0$ are defined on $[0, \infty)$. For example, one can take

$$\eta^\delta(x) := \frac{c}{|x|^n} \exp\left(\frac{-1}{|x|^2 - 1}\right) \quad \text{if} \quad |x| < 1,$$

and the constant $c > 0$ is chosen so that $\int_{\mathbb{R}^n} \eta(x) \, dx = 1$. \hfill (1.12)

Lemma 1.9. Let $u \in C^2(\Omega)$ be a harmonic function in an open set $\Omega \subset \mathbb{R}^n$, and let $\eta^\delta$ be a radially symmetric function satisfying the properties (1.11). Then

$$u(x) \equiv (u * \eta^\delta)(x) := \int_{B_\delta(0)} u(x - y) \eta^\delta(y) \, dy \quad (1.13)$$
on the set $\Omega^\delta := \{x \in \Omega : \text{dist}(x, \partial \Omega) > \delta\}.$

Proof. Fix a point $x \in \Omega^\delta$. Then $B_\delta(x) \subset \Omega$. Using polar coordinates in $B_\delta(0)$, we can write

$$(u * \eta^\delta)(x) = \int_{B_\delta(0)} u(x - y) \eta^\delta(y) \, dy = \delta \int_0^\delta \left[ \int_{\partial B_r(0)} u(x - y) \eta^\delta(y) \, dS_y \right] dr.$$

Note that $\eta^\delta(y) = \eta^\delta_0(|y|)$ on $\partial B_r(0)$, and by virtue of (1.6),

$$\int_{\partial B_r(0)} u(x - y) \, dS_y = \int_{\partial B_r(x)} u(z) \, dS_z = u(x) \cdot |\partial B_r|.$$
Therefore,
\[
(u \ast \eta^\delta)(x) = u(x) \cdot \int_0^\delta \eta^\delta(r) \cdot |\partial B_r| \, dr = u(x) \cdot \int_0^\delta \left[ \int_{\partial B_r(0)} \eta^\delta(y) \, dS_y \right] \, dr
\]

\[
= u(x) \cdot \int_{B_r(0)} \eta^\delta(y) \, dy = u(x)
\]
on the set \( \Omega^\delta \).

**Theorem 1.10 (Smoothness).** Let \( u \in C^2(\Omega) \) be a harmonic function in an open set \( \Omega \subset \mathbb{R}^n \). Then \( u \in C^\infty(\Omega) \).

**Proof.** For \( x \in \Omega \), one can rewrite the equality (1.13) in the form

\[
(1.14) \quad u(x) = \int_{B_\delta(x)} u(y) \eta^\delta(x - y) \, dy = \int_{\mathbb{R}^n} u(y) \eta^\delta(x - y) \, dy.
\]

The integrals here are well defined, because \( B_\delta(x) \subset \Omega \), and \( \eta^\delta(x - y) \equiv 0 \) on \( \mathbb{R}^n \setminus B_\delta(x) \). Since \( \eta^\delta \in C^\infty \), we have \( u \in C^\infty(\Omega^\delta) \) for each \( \delta > 0 \), so that \( u \in C^\infty(\Omega) \). \( \square \)

**Theorem 1.11 (Convergence theorem).** Let \( \{u_m\} \subset C^2(\Omega) \) be a sequence of harmonic functions in an open set \( \Omega \subset \mathbb{R}^n \), which converges in \( L^1(B_r) \) to a function \( u \in L^1_{\text{loc}}(\Omega) \) for each ball \( B_r \subset B_r \subset \Omega \). Then the limit function \( u \) (possibly after redefining on a set of zero measure) belongs to \( C^\infty(\Omega) \) and is harmonic in \( \Omega \).

**Proof.** Fix a ball \( B_r := B_r(x_0) \subset B_r \subset \Omega \). By virtue of (1.14), we have

\[
(1.15) \quad u_m(x) = (u_m \ast \eta^\delta)(x) := \int_{B_\delta(x)} u_m(y) \eta^\delta(x - y) \, dy
\]

for all \( m = 1, 2, \ldots; x \in B_{r/2}(x_0) \), and \( 0 < \delta \leq r/2 \). Therefore,

\[
|(u_i - u_j)(x)| \leq \int_{B_{r/2}(x)} |(u_i - u_j)(y)| \cdot \eta^{r/2}(x - y) \, dy,
\]

and

\[
\sup_{B_{r/2}(x_0)} |u_i - u_j| \leq \sup(\eta^{r/2}) \cdot \int_{B_r(x_0)} |u_i - u_j| \, dy \to 0 \quad \text{as} \quad i, j \to \infty.
\]

This means that \( \{u_m\} \) is a Cauchy sequence in \( C(B_{r/2}(x_0)) \). Since this is true for all \( x_0 \in \Omega \) and \( 0 < r < \text{dist}(x_0, \partial \Omega) \), the sequence \( \{u_m\} \) converges in \( C(\Omega') \) for any bounded open set \( \Omega' \subset \overline{\Omega} \subset \Omega \), and the limit function
1.4. Regularity of harmonic functions

$u$ belongs to $C(\Omega)$. Then from (1.15) it follows that for all $x \in \Omega$ and $0 < \delta < \text{dist}(x, \partial \Omega)$, we have

$$u(x) = \lim_{m \to \infty} u_m(x) = \lim_{m \to \infty} (u_m \ast \eta^\delta)(x) = (u \ast \eta^\delta)(x),$$

which implies $u \in C^\infty(\Omega)$, and also

$$u(x) = \lim_{m \to \infty} (u_m \ast (\Delta \eta^\delta))(x) = \lim_{m \to \infty} ((\Delta u_m) \ast \eta^\delta)(x) = 0.$$

Theorem is proved. \hfill \square

If we take $\eta^\delta(x) := \delta^{-n} \eta(\delta^{-1} x)$ with $\eta(x)$ defined in (1.12), then by virtue of (1.14),

$$D^i u(x) = \delta^{-n-|l|} \int_{B_\delta(0)} u(y) D^i \eta(\delta^{-1}(x - y)) \, dy, \quad x \in \Omega^\delta,$$

for any derivative $D^l u := D^l_1 \cdots D^l_n u(x)$ of order $|l| := l_1 + \cdots + l_n$. In turn, this implies the estimate

$$\max_{|l| = k} \sup_{\Omega^\delta} |D^l u| \leq N_k \delta^{-k} \sup_{\Omega} |u| \quad \text{for} \quad \delta > 0, \quad k = 1, 2, \ldots,$$

where $N_k := \max_{|l| = k} \sup_{\Omega^\delta} |D^l \eta|$. The following theorem contains an improved version of this estimate.

**Theorem 1.12 (Estimates of derivatives).** Let $u \in C^\infty(\Omega)$ be a bounded harmonic function in an open set $\Omega \subset \mathbb{R}^n$. Then the estimate (1.16) holds true with $N_k := (nk)^k$.

**Proof.** We first consider the case $k = 1$. Note that if $u$ is harmonic, then $\Delta(D_i u) = D_i(\Delta u) = 0$, i.e. $D_i u$ is harmonic for each $i = 1, \ldots, n$. Moreover, we can write $D_i u = \text{div}(ue_i)$, where $e_i$ is the $i$-th coordinate vector in $\mathbb{R}^n$. For $x \in \Omega^\delta$, we have $B_\delta(x) \subset B_\delta(0) \subset \Omega$. The mean value formula (1.7), together with the divergence theorem, yield

$$D_i u(x) = \frac{1}{|B_\delta|} \int_{B_\delta(x)} D_i u(y) \, dy = \frac{1}{|B_\delta|} \int_{\partial B_\delta(x)} u(y) e_i \cdot \nu \, dS_y,$$

where $\nu$ is the unit outward vector to $\partial B_\delta(x)$. Hence

$$|D_i u(x)| \leq \frac{|\partial B_\delta|}{|B_\delta|} \sup_{\partial B_\delta(x)} |u| \leq \frac{n}{\delta} \sup_{\Omega} |u|$$

for all $x \in \Omega^\delta$ and $i = 1, \ldots, n$, i.e. we have (1.16) for $k = 1$ with $N_1 = n$. 

In the case $k \geq 2$, we set $r := \delta/k$ and apply the above estimate to higher order derivatives of $u$:

$$\max_{|l|=k} \sup_{\Omega} |D^l u| = \max_{|l|=k} \sup_{\Omega^r} |D^l u| \leq \frac{n}{r} \cdot \max_{|l|=k-1} \sup_{\Omega^{(k-1)r}} |D^l u| \leq \cdots \leq \left( \frac{n}{r} \right)^{k-1} \max_{|l|=1} \sup_{\Omega} |D^l u| \leq \left( \frac{n}{r} \right)^k \sup_{\Omega} |u|.$$ 

Theorem is proved.

\begin{proof}
Fix an arbitrary point $y \in \Omega$. We need to show that

$$u(x) = \sum_{|l|=k} \frac{D^l u(y)}{l!} (x - y)^l$$

in a neighborhood of $y$, or equivalently,

$$u(x) = \sum_{|l| \leq k-1} \frac{D^l u(y)}{l!} (x - y)^l + R_k(x),$$

where $R_k(x) \to 0$ as $k \to \infty$. Note that by the Taylor formula

$$R_k(x) = \sum_{|l|=k} \frac{D^l u(z)}{l!} (x - y)^l,$$

where $z$ belongs to the segment $[y, x]$. Fix a constant $r_0 \in (0, \frac{1}{2} \text{dist}(y, \partial \Omega)$. 

By Theorem 1.12 applied to the ball $B_{2r_0}(y) \subset B_{2r_0}(y) \subset \Omega$, we have

$$\max_{|l|=k} \sup_{B_{r_0}(y)} |D^l u| \leq \left( \frac{n}{r_0} \right)^k M_0, \quad \text{where} \quad M_0 := \sup_{B_{2r_0}(y)} |u|.$$ 

This estimate implies

$$|R_k(x)| \leq \left( \frac{n}{r_0} \right)^k M_0 \cdot \sum_{|l|=k} \frac{|(x - y)^l|}{l!}.$$ 

Note that by the multinomial theorem,

$$(a_1 + \cdots + a_n)^k = \sum_{|l|=k} \frac{k!}{l!} a_1^l,$$

we also have

$$\sum_{|l|=k} \frac{1}{l!} = \frac{n^k}{k!}.$$
Therefore, for $x \in B_r(y)$, $0 < r \leq r_0$, we get
\[ |R_k(x)| \leq M_k := \left( \frac{n^2 kr}{r_0^2} \right)^k \frac{M_0}{k!}, \quad k \geq 1. \]

We see that
\[ \frac{M_{k+1}}{M_k} = \frac{n^2 r}{r_0} \left( 1 + \frac{1}{k} \right)^k \leq \theta := \frac{en^2 r}{r_0}. \]

Finally, we choose $0 < r < \frac{r_0}{en^2}$. Then $0 < \theta < 1$, and by iteration, we obtain $M_k \leq M_0 \theta^k$. This means that $R_k(x) \to 0$ uniformly on the ball $B_r(y)$ as $k \to \infty$, i.e. $u$ is analytic at the point $y$. Since $y$ is an arbitrary point in $\Omega$, the harmonic function $u$ is analytic in $\Omega$.

As a standard property of harmonic functions, the following statement hold true.

**Corollary 1.14** (Uniqueness of continuation). Let $u_1$ and $u_2$ be harmonic functions in an open connected set $\Omega \subset \mathbb{R}^n$, such that $u_1 \equiv u_2$ in a ball $B_r(y) \subset \Omega$. Then $u_1 \equiv u_2$ in $\Omega$.

At the beginning of Section 1.1, we already used the classical divergence theorem in a special case of the vector field $F = Du$. For our reference, we formulate this theorem under general assumptions on $F$ and $\Omega$.

**Theorem 1.15.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with the boundary $\partial \Omega$ of class $C^1$, and let $\mathbf{v}$ be the unit outward normal to $\partial \Omega$. Then for any vector field $F = (F_1, \ldots, F_n) \in C^1(\Omega) \cap C(\overline{\Omega})$, such that $\text{div} F := D_1 F_1 + \cdots + D_n F_n \in L^1(\Omega)$, we have
\[
(1.17) \quad \int_{\Omega} \text{div} F \, dx = \int_{\partial \Omega} F \cdot \mathbf{v} \, dS.
\]

Note that $\partial \Omega \in C^1$ if there exists a function $\Psi \in C^1(\Omega)$ such that
\[
(1.18) \quad \Omega = \{ x \in \mathbb{R}^n : \Psi(x) > 0 \}, \quad \text{and} \quad D \Psi \neq 0 \quad \text{on} \quad \partial \Omega.
\]

Since $\partial \Omega = \{ \Psi = 0 \}$ - a level set of the function $\Psi$, the unit outward normal $\mathbf{v} = -D\Psi / |D\Psi| \in C(\partial \Omega)$, so that $F \cdot \mathbf{v} \in C(\partial \Omega)$.

**Lemma 1.16** (Green’s formula). Let $\Omega$ be as in the previous theorem, and let functions $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be such that $u \Delta v - v \Delta u \in L^1(\Omega)$. Then
\[
(1.19) \quad \int_{\Omega} (u \Delta v - v \Delta u) \, dx = \int_{\partial \Omega} \left( u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) \, dS.
\]

**Proof.** It suffices to apply the divergence theorem to the vector field $F = uDv - vDu$. \qed
Corollary 1.17. Let $u$ be a harmonic function in an arbitrary domain $\Omega \subset \mathbb{R}^n$. Then

$$\int_{\Omega} u \Delta v \, dx = 0 \quad \forall v \in C_0^\infty(\Omega).$$

Proof. By definition, each function $v \in C_0^\infty(\Omega)$ belongs to $C^\infty(\Omega)$, and $v \equiv 0$ on $\Omega \setminus \Omega'$ for some bounded subdomain $\Omega' \subset \overline{\Omega'} \subset \Omega$. Since $\text{dist}(\Omega', \partial\Omega) > 0$, one can always replace $\Omega'$ by another subdomain $\Omega''$ with smooth boundary $\partial\Omega''$, such that $\Omega' \subset \Omega'' \subset \overline{\Omega''} \subset \Omega$. Then $u, v \in C^2(\overline{\Omega''})$, $v = |Dv| = 0$ on $\partial\Omega''$, and the desired equality (1.20) follows from Green's formula applied to the domain $\Omega''$. \hfill \square

By our previous considerations, from $u \in C^2(\Omega)$ and $\Delta u = 0$ in $\Omega$ it follows that $u \in C^\infty(\Omega)$ and $u$ satisfies (1.20). Note that the integral in (1.20) is well-defined for functions $u \in L^1_{\text{loc}}(\Omega)$. It turns out that if $u \in L^1_{\text{loc}}(\Omega)$ satisfies (1.20), then $u \in C^\infty(\Omega)$ and $\Delta u = 0$ in $\Omega$. This is the statement of Weyl's lemma below. For its proof, we need some of the following properties of the operation of convolution.

Proposition 1.18. Let $u$ be a function in $L^1_{\text{loc}}(\Omega)$, and let $\eta^\delta, \delta > 0$ be non-negative functions satisfying (1.11). Then the functions

$$u^{(\delta)}(x) := (u * \eta^\delta)(x) := \int_{B^\delta(y)} u(x - y) \eta^\delta(y) \, dy$$

belong to $C^\infty(\Omega^\delta)$, where $\Omega^\delta := \{ x \in \Omega : \text{dist}(x, \partial\Omega) > \delta \}$, and satisfies the following properties:

(i) For each $x \in \Omega^\delta$,

$$|u^{(\delta)}(x) - u(x)| \leq \omega_x(\delta) := \sup_{|h| \leq \delta} |u(x + h) - u(x)|.$$

In particular, if $u \in C(\Omega)$, then for any bounded subdomain $\Omega' \subset \overline{\Omega'} \subset \Omega$, $u^{(\delta)} \to u$ in $C(\overline{\Omega'})$ as $\delta \to 0^+$:

$$\sup_{\Omega'} |u^{(\delta)} - u| \leq \sup_{x \in \Omega'} \omega_x(\delta) \to 0 \quad \text{as} \quad \delta \to 0^+.$$

(ii) The operation of convolution does not increase the $L^1$-norm:

$$\|u^{(\delta)}\|_{1, \Omega^\delta} \leq \|u\|_{1, \Omega} := \int_{\Omega} |u(x)| \, dx.$$

(iii) For any bounded subdomain $\Omega' \subset \overline{\Omega'} \subset \Omega$, $u^{(\delta)} \to u$ in $L^1(\Omega')$ as $\delta \to 0^+$:

$$\|u^{(\delta)} - u\|_{1, \Omega'} \to 0 \quad \text{as} \quad \delta \to 0^+.$$
1.5. Growth lemma and its application

Lemma 1.19 (Weyl’s lemma). Let $u \in L^1_{loc}(\Omega)$ satisfy the property (1.20). Then $u \in C^\infty(\Omega)$, and $u$ is harmonic in $\Omega$.

**Proof.** Fix $y \in \Omega$ and $r > 0$ such that the ball $B := B_r(y) \subset B_{2r}(y) \subset \Omega$. For $0 < \delta < r$, the functions $u^{(\delta)} \in C^\infty(B)$, and

$$\Delta u^{(\delta)}(x) = (u \ast \eta^{(\delta)})(x) = \int_{B_{\delta}(0)} u(y) \Delta_x \eta^{(\delta)}(x - y) \, dy.$$ 

Here $\Delta_x$ denotes the Laplacian with respect to the variables $x = (x_1, \ldots, x_n)$. It is easy to see that $\Delta_x \eta^{(\delta)}(x - y) \equiv \Delta_y \eta^{(\delta)}(x - y)$. For fixed $x \in B$ and $0 < \delta < r$, the function $v(y) := \eta^{(\delta)}(x - y) \in C^\infty(\Omega)$, so that by virtue of (1.20) we have $\Delta u^{(\delta)}(x) = 0$. Hence $u^{(\delta)}$, $0 < \delta < r$, are harmonic functions on $B$. By Proposition 1.18 (iii), $u^{(\delta)} \to u$ in $L^1(B)$ as $\delta \to 0^+$. Now it remains to apply the convergence Theorem 1.11. \qed

1.5. Growth lemma and its application

**Lemma 1.20** (Growth lemma). Let $\Omega$ be an open set in $\mathbb{R}^n$, and let $x_0 \in \Omega$ and $r > 0$ be such that the measure

$$|\Omega \cap B_r(x_0)| \leq \mu \cdot |B_r|, \quad \mu = \text{const} \in (0, 1).$$ 

Let $v$ be a function in $C^2(\Omega) \cap C(\overline{\Omega})$, such that

$$v \geq 0, \quad \Delta v \geq 0 \quad \text{in} \quad \Omega \cap B_r(x_0); \quad v = 0 \quad \text{on} \quad (\partial \Omega) \cap B_r(x_0).$$

Then

$$v(x_0) \leq \mu \cdot M, \quad \text{where} \quad M := \sup_{\Omega \cap B_r(x_0)} v,$$

and $\mu$ is the constant in (1.22).

**Proof.** Fix $\varepsilon > 0$ and choose a function $\Phi_\varepsilon \in C^\infty(\mathbb{R}^1)$ (depending on $\varepsilon$) such that

$\Phi_\varepsilon, \Phi'_\varepsilon, \Phi''_\varepsilon \geq 0$ on $\mathbb{R}^1$, $\Phi_\varepsilon \equiv 0$ on $(-\infty, \varepsilon]$, and $\Phi'_\varepsilon \equiv 1$ on $[\varepsilon, \infty)$.

It is easy to check that the convolution

$$\Phi_\varepsilon(x) := (\Phi_0 * \eta^\varepsilon)(x - 2\varepsilon), \quad \text{where} \quad \Phi_0(x) := x_+ := \max(x, 0),$$

satisfies all these properties, if we take $\eta(x) := \varepsilon^{-n} \eta(\varepsilon^{-1}x)$, where the function $\eta$ is defined in (1.12). Further, define

$$v_\varepsilon := \Phi_\varepsilon(v) \quad \text{in} \quad \Omega \cap B_r(x_0), \quad v_\varepsilon \equiv 0 \quad \text{on} \quad \overline{B_r(x_0)} \setminus \Omega.$$

From the above properties of the function $\Phi_\varepsilon$ it follows

$$v - 2\varepsilon \leq v_\varepsilon \leq (v - \varepsilon)_+ < M \quad \text{in} \quad \Omega \cap B_r(x_0).$$
Since \( v = 0 \) on the set \( (\partial \Omega) \cap B_r(x_0) \), the function \( v_\varepsilon \) vanishes near this set. Hence we have \( v_\varepsilon \in C^2(B_r(x_0)) \) and \( v_\varepsilon \geq 0 \) in \( B_r(x_0) \). Moreover, \( Dv_\varepsilon = \Phi_\varepsilon'(v) \cdot Dv \), and
\[
\Delta v_\varepsilon = \Phi_\varepsilon''(v) \cdot \Delta v + \Phi_\varepsilon''(v) \cdot |Dv|^2 \geq 0 \quad \text{in} \quad B_r(x_0).
\]
Now we can use the mean value formula (1.7), which gives us
\[
v_\varepsilon(x_0) \leq \frac{1}{|B_r|} \int_{B_r(x_0)} v_\varepsilon(x) \, dx \leq \frac{1}{|B_r|} \int_{\Omega \cap B_r(x_0)} M \, dx \leq \frac{|\Omega \cap B_r(x_0)|}{|B_r|} \cdot M \leq \mu \cdot M.
\]
Since \( v_\varepsilon(x_0) \to v(x_0) \) as \( \varepsilon \to 0^+ \), the desired estimate (1.24) follows.

**Definition 1.21.** An open set \( \Omega \subseteq \mathbb{R}^n \) satisfies the condition \( A \) with a constant \( \mu_0 \in (0, 1) \) if the measure
\[
|\Omega \cap B_r(y)| \leq \mu_0 \cdot |B_r| \quad \text{for all} \quad y \in \partial \Omega \quad \text{and} \quad r > 0.
\]

**Theorem 1.22.** Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \) satisfying the condition \( A \) with a constant \( \mu_0 > 0 \). Let an open subset \( \Omega' \subseteq \Omega \) and a function \( u \in C^2(\Omega') \cap C(\Omega') \) be such that \( u > 0 \) in \( \Omega' \) and \( u = 0 \) on \( \partial \Omega' \). Then there are constants \( \gamma > 0 \) and \( N > 0 \), which depend only on \( n \) and \( \mu_0 \), such that
\[
M := \sup_{\Omega'} d^{-\gamma} u \leq N \cdot F, \quad \text{where} \quad F := \sup_{\Omega'} d^{2-\gamma}(\Delta u)_-,
\]
\( d = d(x) := \text{dist}(x, \partial \Omega), \quad (\Delta u)_- := \max(-\Delta u, 0) \).

**Proof.** Without loss of generality, we may assume that \( \text{dist}(\Omega', \partial \Omega) > 0 \). Indeed, for \( \delta > 0 \), the function \( u - \delta \) and the domain \( \Omega'_\delta := \Omega \cap \{u > \delta\} \) satisfy these assumptions. Therefore, if we first prove the estimate (1.30) for \( u - \delta \) in \( \Omega'_\delta \), then this estimate for the original function \( u \) in \( \Omega' \) follows by taking \( \delta \to 0^+ \).

This additional assumption guarantees that \( d^{-\gamma} \) is bounded in \( \Omega' \) for \( \gamma > 0 \) (the constant \( \gamma \in (0, 1) \) will be specified later). Hence the function \( d^{-\gamma} u \) attains its maximum on \( \Omega' \) at some point \( x_0 \in \Omega' \), i.e.
\[
M = r^{-\gamma} u(x_0), \quad \text{where} \quad r := d(x_0) := \text{dist}(x_0, \partial \Omega) > 0.
\]

We will take into consideration the auxiliary function
\[
w(x) := \frac{1}{2n} (4r^2 - |x - x_0|^2)
\]
which satisfies
\[
w > 0, \quad \Delta w = -1 \quad \text{in} \quad B_{2r}(x_0).
\]
For fixed constants $\gamma, \varepsilon \in (0, 1)$, consider the function
\begin{equation}
\label{1.33}
v(x) := u(x) - (\varepsilon r)^{\gamma} M - (\varepsilon r)^{\gamma - 2} F \cdot w(x).
\end{equation}
First we assume that $v(x_0) > 0$. Then $x_0$ belongs to the set
\begin{equation}
\label{1.34}
V := \Omega' \cap B_{2r}(x_0) \cap \{ x : d(x) > \varepsilon r \} \cap \{ x : v(x) > 0 \},
\end{equation}
so that $V$ is not empty.

We are going to apply the Growth lemma to the function $v$ in the domain $V$, with the ball $B_{2r}(x_0)$ in place of $B_r(x_0)$. Note that $r := \text{dist}(x_0, \partial \Omega) = |x_0 - y_0|$ for some point $y_0 \in \partial \Omega$. Then $B_r(y_0) \subset B_{2r}(x_0)$, and also $V \subseteq \Omega' \subseteq \Omega$. Since $\Omega$ satisfies the condition (A), i.e. inequality (1.29), at the point $y = y_0 \in \partial \Omega$, we get
\[
|B_{2r}(x_0) \setminus V| \geq |B_r(x_0) \setminus \Omega| = |B_r| - |\Omega \cap B_r(y_0)|
\geq (1 - \mu_0) \cdot |B_r| = (1 - \mu_0) 2^{-n} \cdot |B_{2r}|,
\]
\[
|V \cap B_{2r}(x_0)| = |B_{2r} - |B_{2r}(x_0) \setminus V| \leq \mu \cdot |B_{2r}|,
\]
where $\mu := 1 - (1 - \mu_0) 2^{-n} \in (0, 1)$. This means the condition (1.22) is satisfied with this constant $\mu = \mu(n, \mu_0) \in (0, 1)$.

Further, we claim that
\begin{equation}
\label{1.35}
v = 0 \quad \text{on} \quad (\partial V) \cap B_{2r}(x_0).
\end{equation}
Suppose otherwise, i.e. $v(y) > 0$ at some point $y \in (\partial V) \cap B_{2r}(x_0)$. Since $y$ is an interior point of $B_{2r}(x_0)$, we have $w(y) > 0$. Moreover, since $0 < v(y) < u(y)$ and $u = 0$ on $\partial \Omega'$, the point $y$ must be an interior point of $\Omega'$. By definition of $M$, we have $u \leq d^{\gamma} M$, hence
\[
0 < v(y) < u(y) - (\varepsilon r)^{\gamma} M \leq [d^{\gamma}(y) - (\varepsilon r)^{\gamma}] \cdot M,
\]
which implies $d(y) > \varepsilon r$, i.e. $y$ is an interior point of the set $\{ x : d(x) > \varepsilon r \}$. Summarizing these arguments, we see that $y$ is an interior point of $V$, in contradiction with our assumption $y \in \partial V$. This contradiction proves the property (1.35).

Finally, since $d > \varepsilon r$ in $V$ and $\gamma - 2 < 0$, we have
\[
-\Delta u \leq (\Delta u)_- \leq d^{\gamma - 2} F \leq (\varepsilon r)^{\gamma - 2} F \quad \text{in} \quad V,
\]
which in turn implies
\[
\Delta v = \Delta u - (\varepsilon r)^{\gamma - 2} F \cdot \Delta w = \Delta u + (\varepsilon r)^{\gamma - 2} F \geq 0 \quad \text{in} \quad V.
\]
Therefore, we can apply the Growth lemma (Lemma 1.20) to the function $v < u$ in the domain $V \subseteq \Omega' \cap B_{2r}(x_0)$, which yields
\begin{equation}
\label{1.36}
v(x_0) \leq \mu \cdot \sup_{V \cap B_{2r}(x_0)} v \leq \mu \cdot \sup_{\Omega' \cap B_{2r}(x_0)} u.
\end{equation}
Since \( d(x_0) = r \), we also have
\[
d = d(x) \leq |x - x_0| + d(x_0) < 3r, \quad u \leq d^\gamma M \leq (3r)^\gamma M
\]
on the set \( \Omega' \cap B_{2r}(x_0) \). From the previous estimate we now obtain
\[
(1.37) \quad v(x_0) \leq \mu \cdot (3r)^\gamma M.
\]
In the proof of this estimate, we assumed \( v(x_0) > 0 \), which guarantees that \( x_0 \) belongs to the set \( V \) in (1.34). In the case \( v(x_0) \leq 0 \), the estimate (1.37) is obvious, so that it is true without any additional assumptions (for fixed \( \gamma, \varepsilon \in (0, 1) \)).

Having in mind that \( u(x_0) = r^\gamma M \) and \( w(x_0) = 2r^2/n \), by virtue of (1.33) and (1.37), we conclude
\[
v(x_0) = r^\gamma M - (3r)^\gamma M - (3r)^{\gamma-2} F \cdot \frac{2}{n} r^2 \leq \mu \cdot (3r)^\gamma M
\]
or equivalently,
\[
(1.38) \quad (1 - \varepsilon^{\gamma} - 3^\gamma \mu) \cdot M \leq \frac{2}{n} \varepsilon^{\gamma-2} F.
\]
Since \( \mu = \mu(n, \mu_0) < 1 \), now it remains to choose a small constant \( \gamma = \gamma(n, \mu_0) > 0 \), and then a small constant \( \varepsilon = \varepsilon(n, \mu_0) > 0 \) in such a way that the constant \( c_0 = c_0(n, \mu_0) := 1 - \varepsilon^{\gamma} - 3^\gamma \mu \) is strictly positive. Then the desired estimate (1.30) follows immediately with the constant \( N = N(n, \mu_0) := \frac{2}{c_0} \cdot \varepsilon^{\gamma-2} \).

1.6. Model problems for the Laplace operator

In this section, we consider three boundary value problems for Laplace’s equation. They are called model problems here because they serve as a foundation for more general theory of elliptic equations with Hölder coefficients. The corresponding Theorems 1.24–1.26 assert the existence of harmonic functions, satisfying specific boundary conditions for each problem, and having certain regularity properties. First we consider the Dirichlet problem in a ball:
\[
(1.39) \quad \Delta u = f \quad \text{in} \quad B_r := B_r(x_0), \quad u = g \quad \text{on} \quad \partial B_r,
\]
with polynomials \( f \) and \( g \). For \( k = 0, 1, 2, \ldots \), denote \( \mathcal{P}_k \) the collection of all polynomials \( p(x) = p(x_1, \ldots, x_n) \) of degree at most \( k \):
\[
\mathcal{P}_k := \left\{ p = p(x) = \sum_{|l| \leq k} c_l x^l, \quad c_l \text{ constants} \right\}
\]

**Lemma 1.23.** Let polynomials \( f \in \mathcal{P}_k \) and \( g \in \mathcal{P}_{k+2} \) be given. Then for arbitrary \( x_0 \in \mathbb{R}^n \) and \( r > 0 \), the Dirichlet problem (1.39) has a unique solution in \( C^2(B_r) \cap C(B_r) \). This solution belongs to \( \mathcal{P}_{k+2} \) and is represented in the form \( u = g + (r^2 - |x - x_0|^2) h \), where \( h \in \mathcal{P}_k \).
1.6. Model problems for the Laplace operator

Proof. Replacing $x$ by $r^{-1}(x-x_0)$, we may assume without loss of generality $x_0 = 0$, $r = 1$. The uniqueness of solution is contained in Theorem 1.6. In order to prove the existence, we set $v := u - g$, so that (1.39) is reduced to the equivalent problem

$$\Delta v = f_0 \quad \text{in } B_1, \quad v = 0 \quad \text{on } \partial B_1,$$

with $f_0 = f - \Delta g \in \mathcal{P}_k$. Next, consider the linear mapping

$$T p := \Delta ((1 - |x|^2)p) : \mathcal{P}_k \rightarrow \mathcal{P}_k.$$

If $T p = 0$, then $v := (1 - |x|^2)p$ is a solution of the problem (1.40) with $f_0 \equiv 0$. By uniqueness, we have $v \equiv 0$ and $p \equiv 0$. Hence $T$ has the trivial null space: $T^{-1}(0) = \{0\}$. Since the dimension of $\mathcal{P}_k$ is finite, we must have $T(\mathcal{P}_k) = \mathcal{P}_k$. Then $Th = f_0$ for some $h \in \mathcal{P}_k$, and $v = (1 - |x|^2)h$ is a solution of the problem (1.40). Finally, $u = g + v = g + (1 - |x|^2)h$ is a solution of the problem (1.39). \[\square\]

Theorem 1.24. For any continuous function $g$ on $\partial B_r := \partial B_r(x_0)$, the Dirichlet problem

$$\Delta u = 0 \quad \text{in } B_r := B_r(x_0), \quad u = g \quad \text{on } \partial B_r,$$

has a unique classical solution (i.e. solution in $C^2(B_r) \cap C(\overline{B_r})$). This solution belongs to $C^\infty(B_r)$ and satisfies the estimate

$$\max_{|l|=k} \sup_{B_{r-\delta}} |D^l u| \leq N_k \delta^{-k} \sup_{B_r} |u| \quad \text{for } 0 < \delta < r, \quad k = 1, 2, \ldots,$$

with the constant $N_k := (nk)^k$.

Proof. Based on the Weierstrass approximation theorem, we can choose a sequence of polynomials $\{g_m\}$ such that $|g - g_m| \leq 1/m$ on $\partial B_r$ for $m = 1, 2, \ldots$. By Lemma 1.23, there exists a sequence of polynomials $\{u_m\}$ satisfying

$$\Delta u_m = 0 \quad \text{in } B_r, \quad u_m = g_m \quad \text{on } \partial B_r.$$

From the comparison principle (Theorem 1.5) it follows

$$\sup_{B_r} |u_i - u_j| = \sup_{\partial B_r} |g_i - g_j| \leq \sup_{\partial B_r} |g_i - g| + \sup_{\partial B_r} |g_j - g| \leq \frac{1}{i} + \frac{1}{j} \rightarrow 0 \quad \text{as } i, j \rightarrow \infty.$$

Therefore, $\{u_m\}$ is a Cauchy sequence in $C(\overline{B_r})$, so it converges uniformly to a function $u \in C(\overline{B_r})$, which implies the convergence in $L^1(B_r)$. By Convergence theorem (Theorem 1.11), the limit function $u \in C^\infty(B_r) \cap C(\overline{B_r})$ and satisfies (1.41). The uniqueness of solution is guaranteed by Theorem 1.6. \[\square\]
In the remaining part of this section, we consider two problems in a half ball
\[ B_r^+ := B_r^+(x_0) := \{ x = (x_1, \ldots, x_n) \in B_r(x_0) : x_n > 0 \}, \]
where \( x_0 \in \mathbb{R}_0^n := \{ x \in \mathbb{R}^n : x_n = 0 \} \). Note that \( \partial B_r^+ = \Gamma \cup S \), where
\begin{align*}
(1.43) \quad \Gamma &:= B_r \cap \{ x_n = 0 \}, & S &:= (\partial B_r^+) \setminus \Gamma = (\partial B_r) \cap \{ x_n \geq 0 \}.
\end{align*}
In other words, \( \Gamma \) and \( S \) are correspondingly the flat and the spherical portions of \( \partial B_r^+ \), and \( \Gamma \) is open relative to \( \partial B_r^+ \).

**Theorem 1.25.** For any continuous function \( g \) on \( \partial B_r^+ \), such that \( g \equiv 0 \) on \( \Gamma \), the Dirichlet problem
\begin{align*}
(1.44) \quad \Delta u = 0 & \quad \text{in} \quad B_r^+, \quad u = g \quad \text{on} \quad \partial B_r^+,
\end{align*}
has a unique classical solution. This solution belongs to \( C^\infty(B_r \cup \Gamma) \) and satisfies the estimate
\begin{align*}
(1.45) \quad \max_{|\lambda| = k} \sup_{B_{r-\delta}^+} |D^{\lambda} u| \leq N_k \delta^{-k} \sup_{B_r^+} |u| \quad \text{for} \quad 0 < \delta < r, \quad k = 1, 2, \ldots,
\end{align*}
with \( N_k := (nk)^k \).

**Proof.** One can extend the function \( g \) from \( S \) to \( \partial B_r \) as an odd function of \( x_n \). Then
\[ g(x) = g(x_1, \ldots, x_{n-1}, x_n) = -g(x_1, \ldots, x_{n-1}, -x_n) \quad \text{on} \quad \partial B_r. \]
The extended function \( g \) is continuous on \( \partial B_r \), so that by the previous theorem, the problem (1.41) has a unique solution \( u \in C^\infty(B_r) \cap C(\overline{B_r}) \).
Then the function
\[ v(x) = v(x_1, \ldots, x_{n-1}, x_n) := -u(x_1, \ldots, x_{n-1}, -x_n) \]
satisfies both the equation \( \Delta v = -\Delta u = 0 \) in \( B_r \) and the boundary condition \( v = g \) on \( \partial B_r \). By uniqueness, we must have \( u \equiv v \) in \( B_r \), which implies \( u \equiv 0 \) on \( \Gamma \). This means that \( u \) is also a solution of the problem (1.44), and the estimate (1.45) follows from (1.42).

The next theorem deals with different kinds of conditions on different parts on \( \partial B_r^+ \). These are the **mixed** boundary conditions
\begin{align*}
(1.46) \quad D_n u = 0 & \quad \text{on} \quad \Gamma, \quad u = g \quad \text{on} \quad S,
\end{align*}
where \( \Gamma \) and \( S \) are defined in (1.43).

**Theorem 1.26.** For any continuous function \( g \) on \( S \), the equation
\( \Delta u = 0 \) in \( B_r^+ \), with the boundary conditions (1.46), has a unique solution \( u \in C^\infty(B_r \cup \Gamma) \cap C(\overline{B_r^+}) \). This solution satisfies the estimate (1.45).
Proof. Now we extend the function \( g \) from \( S \) to \( \partial B_r \) as an even function of \( x_n \):

\[
g(x) = g(x_1, \ldots, x_{n-1}, x_n) = g(x_1, \ldots, x_{n-1}, -x_n) \quad \text{on} \quad \partial B_r.
\]

As in the previous proof, the problem (1.41) has a unique solution \( u \in C^\infty(B_r) \cap C(\overline{B_r}) \). By uniqueness,

\[
u(x) \equiv v(x) := u(x_1, \ldots, x_{n-1}, -x_n) \quad \text{in} \quad B_r.
\]

From this equality it follows

\[
D_n u = D_n v = -D_n u \quad \text{on} \quad \Gamma := B_r \cap \{x_n = 0\},
\]

so that \( D_n u = 0 \) on \( \Gamma \). Therefore, the function \( u \) satisfies all the desired properties in this theorem.

Now it remains to show that the solution is unique. If \( u_1 \) and \( u_2 \) are two harmonic functions in \( B_r^+ \), both satisfying (1.46), then \( u := u_1 - u_2 \) satisfies (1.46) with \( g = 0 \). Hence we need to show that from \( D_n u = 0 \) on \( \Gamma \) and \( u = 0 \) on \( S \) it follows \( u = 0 \) in \( B_r^+ \). Note that for any \( \varepsilon > 0 \), the function \( u^\varepsilon := u + \varepsilon x_n \) is harmonic in \( B_r^+ \) and belongs to \( C^\infty(B_r) \cap C(\overline{B_r}) \). By the maximum principle (Theorem 1.3) it attains its maximum on \( \overline{B_r^+} \) at some point \( y \in \partial B_r^+ = \Gamma \cup S \). However, since \( D_n u^\varepsilon = D_n u + \varepsilon = \varepsilon > 0 \) on \( \Gamma \), the point \( y \) cannot belong to \( \Gamma \). Then \( y \in S \), and

\[
\sup_{B_r^+} u^\varepsilon = \sup_S u^\varepsilon = \sup_S (\varepsilon x_n) = \varepsilon r.
\]

This implies that \( u = \lim_{\varepsilon \to 0^+} u^\varepsilon \leq 0 \) in \( B_r^+ \). Quite similarly (or replacing \( u \) by \( -u \)), we also get \( u \geq 0 \) in \( B_r^+ \). Therefore, \( u \equiv 0 \), and the uniqueness follows. \( \Box \)

1.7. Removable singularities for harmonic functions

For radially symmetric functions \( v(x) := v_0(r) \), \( r := |x| \), the Laplace’s operator has the form

\[
\Delta v(x) = v_0''(r) + \frac{n-1}{r} v_0'(r) = r^{1-n} \left( v_0''(r) \right)'.
\]

Solving a simple ODE, one can describe all harmonic functions \( h(x) := h_0(r) \) on \( \mathbb{R}^n \setminus \{0\} \) as follows

\[
h(x) = \begin{cases} c_1 + c_2 \cdot \ln |x|, & \text{if } n = 2; \\
c_1 + c_2 \cdot |x|^{2-n}, & \text{if } n \neq 2,
\end{cases}
\]

where \( c_1 \) and \( c_2 \) are arbitrary constants.
Theorem 1.27 (Removable singularity). Let \( \Omega \) be an open set in \( \mathbb{R}^n \), and let \( u \) be a harmonic function in \( \Omega \setminus \{x_0\} \) for some point \( x_0 \in \Omega \), such that

\[
(1.49) \quad \lim_{x \to x_0} \frac{u(x)}{h(x - x_0)} = 0, \quad \text{where} \quad h(x) := \begin{cases} -\ln |x|, & \text{if } n = 2; \\ |x|^{2-n}, & \text{if } n \neq 2, \end{cases}
\]

Then \( u \) can be extended as a harmonic function to the whole set \( \Omega \).

**Proof.** Without loss of generality, we can assume \( x_0 = 0 \). Choose \( r \in (0, 1) \) small enough, so that \( B_r := B_r(0) \subset \overline{B_r} \subset \Omega \). By Theorem 1.24, there exists a solution \( \tilde{u} \in C^2(B_r) \cap C(\overline{B_r}) \) of the problem

\[
\Delta \tilde{u} = 0 \quad \text{in} \quad B_r, \quad \tilde{u} = u \quad \text{on} \quad \partial B_r.
\]

Next, for \( 0 < \delta < r \), we compare \( u(x) \) with the functions

\[
w_{1,2}^\delta(x) := \tilde{u}(x) \pm \beta(\delta) \cdot h(x) \quad \text{on} \quad \Omega^\delta := B_r \setminus \overline{B_\delta},
\]

where

\[
0 \leq \beta(\delta) := \sup_{\partial B_\delta} \frac{|u(x) - \tilde{u}(x)|}{h(x)} \to 0 \quad \text{as} \quad \delta \to 0^+
\]

by the assumption (1.49). All three functions \( u, w_1^\delta, w_2^\delta \) are harmonic in \( \Omega^\delta \), and \( w_1^\delta \geq u \geq w_2^\delta \) on \( \partial \Omega^\delta \). By the comparison principle, we have \( w_1^\delta \geq u \geq w_2^\delta \) in \( \Omega^\delta \). Then for arbitrary \( x \in B_r \setminus \{0\} \),

\[
|u(x) - \tilde{u}(x)| \leq \beta(\delta) \cdot h(x) \to 0 \quad \text{as} \quad \delta \to 0^+.
\]

Therefore, \( u \equiv \tilde{u} \) on \( B_r \setminus \{0\} \), and the function \( u \), extended as \( u(0) := \tilde{u}(0) \), is harmonic in \( \Omega \).

**Remark 1.28.** It follows from this theorem that the Dirichlet problem

\[
(1.50) \quad \Delta u = 0 \quad \text{in} \quad \Omega := B_1 \setminus \{0\}; \quad u = 0 \quad \text{on} \quad \partial B_1, \quad u(0) = 1;
\]

does not have solutions in \( C^2(\Omega) \cap C(\overline{\Omega}) \).

One can use similar arguments in order to “fix” boundary singularities. Namely, the maximum principle (Theorem 1.3) and the comparison principle (Theorem 1.5) are applied to functions in \( C^2(\Omega) \cap C(\overline{\Omega}) \), where \( \Omega \) is a bounded open set in \( \mathbb{R}^n \). These statements may fail if we allow discontinuity even at a single point \( x_0 \in \partial \Omega \).

**Example 1.29.** Note that any derivative of a harmonic function is harmonic. Differentiating the function \( h(x) \) in (1.48), we get:

\[
(1.51) \quad h_0(x) := \frac{x_n}{|x|^n} = \text{const} \cdot D_n h(x) \quad \text{is harmonic in} \quad \mathbb{R}^n \setminus \{0\}.
\]
If we take \( \Omega := B^+_1(0) := \{ x \in \mathbb{R}^n : |x| < 1, x_n > 0 \} \), and \( x_0 := 0 \in \partial \Omega \), then \( h_0 \) is harmonic in \( \Omega \), belongs to \( C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\}) \), and does not satisfy the maximin principle:

\[
\infty = \sup_{\Omega} h_0 > \sup_{(\partial \Omega) \setminus \{0\}} h_0 = 1.
\]

In this example, the harmonic function \( h_0 \) has singularity of order \( |x|^{1-n} \) at the point \( x_0 = 0 \in \partial \Omega \). The following theorem states that the maximum principle, and hence the comparison principle, are applicable for weaker singularities.

**Theorem 1.30.** Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n, n \geq 2 \), such that

\[
\Omega \cap B_r(0) = B^+_r := \{ x \in \mathbb{R}^n : |x| < r, x_n > 0 \}
\]

for some \( r > 0 \). Let \( u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\}) \) be a harmonic function in \( \Omega \) satisfying the property

\[
u(x) \cdot |x|^{n-1} \to 0 \quad \text{as} \quad x \to 0.
\]

Then

\[
M := \sup_{\Omega} u = M_0 := \sup_{(\partial \Omega) \setminus \{0\}} u.
\]

**Proof.** Denote

\[
m_r := \sup_{S_r} u, \quad \text{where} \quad S_r := \{ x \in \mathbb{R}^n : |x| = r, x_n > 0 \} \subset \partial B^+_r.
\]

First we reduce the proof to the case \( \Omega = B^+_r \), i.e.

\[
\sup_{B^+_r} u = \sup_{(\partial B^+_r) \setminus \{0\}} u.
\]

Indeed, the set \((\partial B^+_r) \setminus \{0\}\) is contained in the union of two sets: \((\partial \Omega) \setminus \{0\}\) and \( S_r \), so that from (1.55) it follows

\[
\sup_{B^+_r} u \leq \max(M_0, m_r).
\]

Moreover, the boundary of \( \Omega \setminus B^+_r \) is also contained in the union of these two sets, hence by the maximum principle,

\[
\sup_{\Omega \setminus B^+_r} u = \sup_{(\partial (\Omega \setminus B^+_r))} u \leq \max(M_0, m_r).
\]

Since we always have \( M_0 \leq M \) and \( m_r \leq M \), these inequalities imply

\[
M := \sup_{\Omega} u = \max(M_0, m_r).
\]

It is easy to see that the case \( M > M_0 \) is impossible: in this case we must have \( M = m_r \), and the maximum of \( u \) would be attained at some point
1. Laplace’s Equation

\( x_0 \in S_r \subset \Omega \), which contradicts to the strong maximum principle. We have proved that from (1.55) it follows (1.53).

Now it remains to prove (1.55), i.e. (1.53) in the case \( \Omega := B^+_r \). Replacing \( u \) by \( u - \text{const} \) is necessary, we may assume \( M_0 = 0 \). Since the function \( h_0 \) in (1.51) is harmonic in \( \mathbb{R}^n \setminus \{0\} \), the functions

\[ h_\delta(x) = h_\delta(x', x_n) := h_0(x', x_n + \delta) = \frac{x_n + \delta}{[|x'|^2 + (x_n + \delta)^2]^{n/2}}, \quad 0 < \delta \leq r, \]

belong to \( C^\infty(B^+_r) \) and satisfy

\[
\begin{align*}
    h_\delta &> 0, \quad \Delta h_\delta = 0 \quad \text{in} \quad B^+_r; \\
    2^{-n} \delta^{1-n} &\leq h_\delta \leq 2 \delta^{1-n} \quad \text{in} \quad B^+_\delta.
\end{align*}
\]

By virtue of (1.52),

\[
\beta(\delta) := \sup_{S_\delta} \frac{u(x)}{h_\delta(x)} \to 0 \quad \text{as} \quad \delta \to 0^+
\]

For each \( \delta \in (0, r] \), the function

\[
u_\delta(x) := u(x) - \beta(\delta) \cdot h_\delta(x)
\]

is harmonic in \( \Omega_\delta := B^+_r \setminus B^+_{\delta} \). By definition of \( \beta(\delta) \), \( u_\delta \leq 0 \) on \( S_\delta \). On the remaining part of \( \partial \Omega_\delta \), which is a subset of \( (\partial B^+_r) \setminus \{0\} \), we have \( u_\delta \leq u \leq M_0 = 0 \). By the maximum principle,

\[
u_\delta \leq \sup_{\partial \Omega_\delta} u \leq 0 \quad \text{in} \quad \Omega_\delta = B^+_r \setminus \overline{B^+_{\rho}}.
\]

For arbitrary \( x \in B^+_r \), we obtain

\[
u(x) = \lim_{\delta \to 0^+} \nu_\delta(x) \leq 0.
\]

This means \( M \leq 0 = M_0 \). Since the opposite inequality is always true, the estimate (1.53) is proved.

\textbf{Corollary 1.31.} In addition to the assumptions of the previous theorem, suppose that \( u = g \) on \( (\partial \Omega) \setminus \{0\} \), where \( g \in C(\partial \Omega) \). Then \( u \) can be extended as a continuous function from \( \overline{\Omega} \setminus \{0\} \) to \( \overline{\Omega} \) by setting \( u(0) := g(0) \).

\textbf{Proof.} By defining

\[
g := u \quad \text{on} \quad S_r := \{|x| = x_n > 0\} = (\partial B^+_r) \setminus (\partial \Omega),
\]

we extend \( g \) as a continuous function on \( \partial B^+_r \). Consider the Dirichlet problem

\[
\Delta \tilde{u} = 0 \quad \text{in} \quad B^+_r, \quad \tilde{u} = g \quad \text{on} \quad \partial B^+_r.
\]
1.8. Dirichlet problem for Laplace’s equation in general domain

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) satisfying the condition (A) in Definition 1.21 with a constant \( \mu_0 \in (0, 1) \). In this section, we prove the existence of a classical solution, i.e. solution in the class \( C^2(\Omega) \cap C(\overline{\Omega}) \), to the Dirichlet problem

\[ (1.57) \quad \Delta u = 0 \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial \Omega, \]

In the next section, we prove that this problem has a classical solution \( \tilde{u} \in C^2(B_r^+) \cap C(B_r^+) \). By the previous theorem, the functions \( v := \pm (\tilde{u} - u) \) satisfy

\[ \sup_{B_r^+ \setminus \{0\}} v = \sup_{\partial B_r^+ \setminus \{0\}} u = 0. \]

Therefore, \( u \equiv \tilde{u} \) on \( (B_r^+) \setminus \{0\} \), and the extended function \( u \) coincides with \( \tilde{u} \in C(B_r^+) \) on \( B_r^+ \).

**Remark 1.32.** A bit longer argument helps to “bypass” the existence result for the problem (1.56) (which is not proved yet), when \( g \) is an arbitrary continuous function on \( \partial B_r^+ \). We do have the existence theorem (Theorem 1.25) in the case \( g \equiv 0 \) on \( \Gamma_r := \{|x| \leq r, x_n = 0\} \) - the flat portion of \( \partial B_r^+ \). Replacing \( u, g \) by \( u - c, g - c \), we easily extend this theorem to the case \( g \equiv c = \text{const} \) on \( \Gamma_r \). For \( 0 < \rho \leq r \), we can now define \( \tilde{u}_\rho \in C^2(B^+_{\rho}) \cap C(B^+_{\rho}) \) as a solution to the problem

\[ \Delta \tilde{u}_\rho = 0 \quad \text{in} \quad B^+_{\rho}, \quad \tilde{u}_\rho = g_\rho \quad \text{on} \quad \partial B^+_{\rho}, \]

where

\[ r_\rho := c_\rho := \sup_{\Gamma_\rho} g, \quad g_\rho := \max(u, c_\rho) \quad \text{on} \quad S_\rho. \]

By Theorem 1.30,

\[ \sup_{B^+_{\rho}} (u - \tilde{u}_\rho) = \sup_{\partial B^+_{\rho} \setminus \{0\}} (u - \tilde{u}_\rho) \leq 0, \quad 0 < \rho \leq r. \]

Then

\[ \limsup_{x \to 0} u(x) \leq \lim_{x \to 0} \tilde{u}_\rho = c_\rho, \quad 0 < \rho \leq r. \]

Here \( c_\rho \to g(0) \) as \( \rho \to 0^+ \). Therefore,

\[ \limsup_{x \to 0} u(x) \leq g(0). \]

Similarly (or replacing \( u \) by \( -u \) and \( g \) by \( -g \)), we also get

\[ \liminf_{x \to 0} u(x) \geq g(0). \]

From here the desired property follows: \( u(x) \to g(0) \) as \( x \to 0 \).
where \( g \) is a continuous function on \( \partial \Omega \). By results in Section A.1, we can extend \( u \) as a continuous function to the whole space \( \mathbb{R}^n \), so that we can assume \( g \in C(\mathbb{R}^n) \).

1.8.1. Existence of a weak solution. First we assume \( g \in C^1(\mathbb{R}^n) \). Consider the energy functional

\[
E[v] := \frac{1}{2} \int_{\Omega} |Dv|^2 \, dx
\]

on the class of “admissible” functions

\[ A := \{ v \in C^1(\overline{\Omega}) : v = g \text{ on } \partial \Omega \}. \]

This class is not empty, because it contains the function \( g \). Denote

\[
m := \inf_{v \in A} E[v] \geq 0,
\]

and choose a sequence \( u_1, u_2, \ldots \), such that

\[
m_k := E[u_k] \to m \quad \text{as} \quad k \to \infty.
\]

For each pair \( i, j \), the function \( \frac{1}{2}(u_i + u_j) \) belongs to \( A \), which implies

\[
m \leq E\left[ \frac{1}{2}(u_i + u_j) \right] = \frac{1}{8} \int_{\Omega} |D(u_i + u_j)|^2 \, dx.
\]

Using the parallelogram identity

\[
|D(u_i + u_j)|^2 + |D(u_i - u_j)|^2 = 2 |Du_i|^2 + 2 |Du_j|^2,
\]

we can write

\[
\int_{\Omega} |D(u_i - u_j)|^2 \, dx = 4 E[u_i] + 4 E[u_j] - 8 E\left[ \frac{1}{2}(u_i + u_j) \right] \leq 4m_i + 4m_j - 8m \to 0 \quad \text{as} \quad i, j \to \infty.
\]

Since \( u_i - u_j = 0 \) on \( \partial \Omega \), by Poincaré inequality,

\[
\int_{\Omega} |u_i - u_j|^2 \, dx \to 0 \quad \text{as} \quad i, j \to \infty.
\]

Therefore, there is a function \( u \in L^2(\Omega) \subset L^1(\Omega) \) such that \( u_k \to u \) in \( L^2(\Omega) \) as \( k \to 0 \), which implies the convergence in \( L^1(\Omega) \):

\[
||u_k - u||_1 := \int_{\Omega} |u_k - u| \, dx \to 0 \quad \text{as} \quad k \to \infty.
\]
Further, for arbitrary function $v \in C_0^\infty(\Omega)$ and $\tau \in \mathbb{R}^1$, the functions $u_k + \tau v$ belong to $\mathcal{A}$, hence

$$m \leq E[u_k + \tau v] = m_k + 2\tau \int_\Omega Du_k \cdot Dv \, dx + \tau^2 E[v]$$

$$= m_k - 2\tau \int_\Omega u_k \cdot \Delta v \, dx + \tau^2 E[v].$$

Taking the limit as $k \to \infty$, we obtain

$$0 \leq -2\tau \int_\Omega u \cdot \Delta v \, dx + \tau^2 E[v] \quad \text{for all} \quad \tau \in \mathbb{R}^1,$$

which is only possible if

$$\int_\Omega u \cdot \Delta v \, dx = 0, \quad v \in C_0^\infty(\Omega).$$

By Weyl’s lemma (Lemma 1.19) we conclude that $u \in C^\infty(\Omega)$, and $\Delta u = 0$ in $\Omega$.

**1.8.2. Comparison principle.** We still need to show that under appropriate assumptions on $\Omega$, the function $u$ belongs to $C(\overline{\Omega})$, and $u = g$ on $\partial \Omega$. Suppose this is the case. Then by the comparison principle, from

(1.62) \quad \text{$w \in C^2(\overline{\Omega})$, $\Delta w < 0$ on $\overline{\Omega}$, and $w \geq g$ on $\partial \Omega$,}

it follows $u \leq w$ in $\Omega$. It turns out that the function $u$ in (1.61) satisfies this property, without any additional assumptions on $\Omega$.

**Lemma 1.33.** Let $w$ be a function satisfying (1.62). Then $u \leq w$ in $\Omega$.

**Proof.** Fix $\varepsilon > 0$, and let $\Phi_\varepsilon \in C^\infty(\mathbb{R}^1)$ be a function in the proof of Growth lemma (Lemma 1.20). Since $u_k \in C^1(\overline{\Omega})$ and $u_k = g \leq w$ on $\partial \Omega$, the functions $u_k := \Phi_\varepsilon(u_k - w)$ belong to $C_0^1(\Omega)$. By the chain rule,

$$Dv_k = \Phi_\varepsilon'(u_k - w) \cdot D(u_k - w).$$

Since $0 \leq \Phi_\varepsilon' \leq 1$, we have

$$0 \leq \int_\Omega |Dv_k|^2 \, dx = \int_\Omega (\Phi_\varepsilon')^2 \cdot |D(u_k - w)|^2 \, dx$$

$$\leq \int_\Omega \Phi_\varepsilon' \cdot |D(u_k - w)|^2 \, dx = \int_\Omega Dv_k \cdot D(u_k - w) \, dx$$

$$= A_k + B_k,$$
where
\[ A_k := \int_{\Omega} Dv_k \cdot D(u_k - u) \, dx, \quad B_k := \int_{\Omega} Dv_k \cdot D(u - w) \, dx. \]

From the convergence in (1.60) it follows that \( Du \in L^2(\Omega) \), and
\[ \int_{\Omega} |D(u_k - u)|^2 \, dx \to 0 \quad \text{as} \quad k \to \infty. \]

By the Cauchy-Schwartz inequality, we obtain
\[ A_k^2 \leq \int_{\Omega} |Dv_k|^2 \, dx \cdot \int_{\Omega} |D(u_k - u)|^2 \, dx \leq \text{const} \cdot \int_{\Omega} |D(u_k - u)|^2 \, dx \to 0 \quad \text{as} \quad k \to \infty. \]

Further, since \( v_k \in C^1_0(\Omega) \), we have
\[ B_k = -\int_{\Omega} v_k \Delta(u - w) \, dx = \int_{\Omega} v_k \Delta w \, dx. \]

From (1.61) it follows
\[ ||v_k - v||_1 \to 0 \quad \text{as} \quad k \to \infty, \quad \text{where} \quad v := \Phi_\varepsilon(u - w) \geq 0. \]

This implies
\[ 0 \leq \lim_{k \to 0} (A_k + B_k) = \int_{\Omega} v \cdot \Delta w \, dx. \]

Since \( v \geq 0, \Delta w < 0 \) in \( \Omega \), and \( v \in C(\Omega) \), we must have
\[ v = \Phi_\varepsilon(u - w) \equiv 0 \quad \text{in} \quad \Omega. \]

This inequality is true for arbitrary \( \varepsilon > 0 \). Hence
\[ (u - w)_+ = \lim_{\varepsilon \to 0^+} \Phi_\varepsilon(u - w) \equiv 0 \quad \text{in} \quad \Omega, \]

i.e. \( u \leq w \) in \( \Omega \).

1.8.3. **Existence in the case** \( \partial \Omega \in C^2, \ g \in C^2 \). Now we assume that the domain \( \Omega \) has boundary of class \( C^2 \). Then it satisfies the following **exterior sphere condition** with a constant \( r_0 > 0 \):

(S) For each point \( x_0 \in \partial \Omega \), there is a ball \( B := B_{r_0}(y_0) \) such that
\[ B \cap \Omega = \emptyset, \quad x_0 \in \overline{B} \cap \overline{\Omega}. \]
Lemma 1.34. Let $\Omega$ satisfy the above condition (S), and let $g \in C^2(\mathbb{R}^n)$. Then the weak solution $u$ satisfies

$$
\lim_{x \to x_0} u(x) = g(x) \quad \text{for each } x_0 \in \partial \Omega.
$$

Therefore, the function $u$, extended as $u := g$ on $\partial \Omega$, belongs to $C^\infty(\Omega) \cap C(\overline{\Omega})$, and solves the problem (1.57) in a classical sense.

**Proof.** Fix constants $\gamma > n - 2 \geq 0$ and $R > \text{diam} (\Omega) + r_0$, where $r_0 > 0$ is a constant in the condition (S). By virtue of this condition, for each $x_0 \in \partial \Omega$, there exists $y_0 \in \mathbb{R}^n$ such that

$$
\Omega \subset \{ x \in \mathbb{R}^n : r_0 < |x - y_0| < R \}, \quad \text{and} \quad x_0 \in \partial B_{r_0}(y_0).
$$

Note that

$$
\Delta (|x|^{-\gamma}) = \gamma (\gamma - n + 2) \cdot |x|^{-\gamma - 2} > 0 \quad \text{for} \quad x \neq 0.
$$

For fixed $x_0 \in \partial \Omega$, set

$$
w(x) := g(x) + N \cdot \left[ r_0^{-\gamma} - |x - y_0|^{-\gamma} \right], \quad \text{where} \quad N = \text{const} > 0.
$$

We have $w \in C^\infty(\overline{\Omega})$, $w \geq g$ on $\partial \Omega$, and

$$
\Delta w(x) = \Delta g(x) - N \gamma (\gamma - n + 2) \cdot |x - y_0|^{-\gamma - 2} \
\leq \sup_{\Omega} (\Delta g) - N \gamma (\gamma - n + 2) \cdot R^{-\gamma - 2} < 0 \quad \text{on} \quad \overline{\Omega},
$$

provided the constant $N > 0$ is large enough. Then from Lemma 1.33 it follows $u \leq w$ in $\Omega$, therefore,

$$
\limsup_{x \to x_0} u(x) \leq \lim_{x \to x_0} w(x) = g(x_0).
$$

Replacing $u$ by $-u$ and $g$ by $-g$, one can also get a similar estimate $\liminf u(x) \geq g(x_0)$ as $x \to x_0$. From these estimates, the equality (1.63) follows. \qed

1.8.4. Existence in the general case.

**Theorem 1.35.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ satisfying the condition (A). Then for any function $g \in C(\mathbb{R}^n)$, the Dirichlet problem (1.57) has a unique solution $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$.

**Proof.** We first assume $g \in C^2(\mathbb{R}^n)$. For small $\varepsilon > 0$, choose open sets $\Omega_\varepsilon$ satisfying the properties

$$
\partial \Omega_\varepsilon \in C^2, \quad \text{and} \quad (\Omega \cap \{d > \varepsilon\}) \subset \Omega_\varepsilon \subset \overline{\Omega_\varepsilon} \subset \Omega,
$$

where $d = d(x) := \text{dist}(x, \partial \Omega)$. By Lemma 1.34, the problems

$$
\Delta u_\varepsilon = 0 \quad \text{in} \quad \Omega_\varepsilon, \quad u_\varepsilon = g \quad \text{on} \quad \partial \Omega_\varepsilon,
$$

have solutions $u_\varepsilon \in C^\infty(\Omega_\varepsilon) \cap C(\overline{\Omega_\varepsilon})$. Since $g \in C(\mathbb{R}^n)$, we have $g \in C^\infty(\partial \Omega) \cap C(\overline{\Omega})$, and $u_\varepsilon \to g$ uniformly on $\Omega_\varepsilon$. Therefore, $u_\varepsilon$ is a solution of (1.57) in $\Omega_\varepsilon$. By the Arzela-Ascoli theorem, there exists a subsequence $u_{\varepsilon_k} \to u$ uniformly on $\Omega$. Since $u_{\varepsilon_k}$ is a solution of (1.57) in $\Omega_\varepsilon_k$, we have $u_{\varepsilon_k} \to u$ uniformly on $\Omega$, and $u$ is a solution of (1.57) in $\Omega$. \qed
have classical solutions \( u_\varepsilon \in C^2(\Omega_\varepsilon) \cap C(\bar{\Omega}_\varepsilon) \). Since we assume \( g \in C^2(\mathbb{R}^n) \), the functions \( v_\varepsilon := u_\varepsilon - g \) satisfy
\[
\Delta v_\varepsilon = f := -\Delta g \quad \text{in} \quad \Omega_\varepsilon, \quad u_\varepsilon = 0 \quad \text{on} \quad \partial \Omega_\varepsilon.
\]

From Theorem 1.22, applied to the functions \( \pm v_\varepsilon \) on \( \Omega' := \Omega_\varepsilon \), it follows the estimate
\[
\sup_{\Omega_\varepsilon} d^{-\gamma}|v_\varepsilon| \leq N \cdot \sup_{\Omega} d^{2-\gamma}|\Delta g| =: N_0,
\]
where the constants \( \gamma > 0 \) and \( N > 0 \) depend only on \( n \) and \( \mu_0 \) in (1.29).

Set \( v := 0 \) on \( \Omega \setminus \Omega_\varepsilon \). Then \( v \in C(\Omega) \), and
\[
\Delta v(x) \leq N_0 d^\gamma(x) \quad \text{in} \quad \Omega,
\]
uniformly with respect to \( \varepsilon > 0 \).

Fix \( \varepsilon_0 > 0 \). Then for \( 0 < \varepsilon_1, \varepsilon_2 < \varepsilon_0 \), we have
\[
\Delta(v_{\varepsilon_1} - v_{\varepsilon_2}) = \Delta(u_{\varepsilon_1} - u_{\varepsilon_2}) \quad \text{in} \quad \Omega \cap \{d > \varepsilon_0\}.
\]
The maximum principle together with (1.67) imply
\[
\sup_{\Omega} |v_{\varepsilon_1} - v_{\varepsilon_2}| = \sup_{\Omega \cap \{d \leq \varepsilon_0\}} |v_{\varepsilon_1} - v_{\varepsilon_2}| \leq 2N_0 d^\gamma(x) \quad \text{for} \quad 0 < \varepsilon_1, \varepsilon_2 < \varepsilon_0.
\]
Therefore,
\[
\limsup_{\varepsilon_1, \varepsilon_2 \to 0^+} \sup_{\Omega} |v_{\varepsilon_1} - v_{\varepsilon_2}| \leq 2N_0 d^\gamma(x).
\]

Here the left side does not depend on \( \varepsilon_0 > 0 \), which can be made arbitrary small. This means \( v_\varepsilon \) converge in \( C(\bar{\Omega}) \) to a function \( v \in C(\bar{\Omega}) \) as \( \varepsilon \to 0^+ \).

Equivalently, \( u_\varepsilon \) extended as \( u := g \) on \( \bar{\Omega} \setminus \Omega_\varepsilon \), converge in \( C(\bar{\Omega}) \) to \( u := v + g \in C(\bar{\Omega}) \), By the convergence theorem (Theorem 1.11), from
\[
\Delta u_\varepsilon = 0 \quad \text{in} \quad \Omega \cap \{d \geq \varepsilon_0\}, \quad 0 < \varepsilon < \varepsilon_0,
\]
it follows that the limit function \( u \) is harmonic in \( \Omega \cap \{d \geq \varepsilon_0\} \). Since \( \varepsilon_0 > 0 \) is arbitrary, the function \( u \) is harmonic in \( \Omega \) and belongs to \( C^\infty(\Omega) \cap C(\bar{\Omega}) \).

This completes the proof in the case \( g \in C^2(\mathbb{R}^n) \).

In the general case, when the function \( g \) belongs to \( C(\mathbb{R}^n) \), we can assume \( u \in C^0(\mathbb{R}^n) \), and then approximate it uniformly on \( \mathbb{R}^n \) by standard convolutions \( g(\delta) \in C^\infty(\mathbb{R}^n) \):
\[
\sup_{\mathbb{R}^n} |g - g(\delta)| \to 0 \quad \text{as} \quad \delta \to 0^+.
\]

For each \( \delta > 0 \), there exists a solution \( u^\delta \in C^\infty(\Omega) \cap C(\bar{\Omega}) \) to the problem
\[
\Delta u^\delta(\delta) = 0 \quad \text{in} \quad \Omega, \quad u = g(\delta) \quad \text{on} \quad \partial \Omega.
\]

By the comparison principle, \( u^\delta \) converges in \( C(\bar{\Omega}) \) to a function \( u \in C(\bar{\Omega}) \) as \( \delta \to 0^+ \). By Theorem 1.11, the limit function \( u \in C^\infty \cap C(\bar{\Omega}) \) is harmonic in \( \Omega \). \qed
1.9. Fundamental solution

1.9.1. Properties of fundamental solution. Changing the order of integration in (1.5) yields

\[ m(r) - u(y) = \frac{1}{\sigma_n} \int_{B_r(y)} \left[ \int_{|x-y|}^r \rho^{1-n} \, d\rho \right] \Delta u(x) \, dx \]

(1.68)

\[ = \int_{B_r(y)} [\Gamma_0(r) - \Gamma_0(|x-y|)] \Delta u(x) \, dx, \]

where

(1.69) \[ \Gamma_0(r) = \begin{cases} \frac{1}{(2-n)\sigma_n} r^{2-n}, & n \neq 2; \\ \frac{1}{2n} \ln r, & n = 2. \end{cases} \]

Definition 1.36. The function \( \Gamma(x) := \Gamma_0(|x|) \), \( x \neq 0 \), is called the fundamental solution of Poisson’s equation \( \Delta u = f \).

Note that for \( x \neq 0 \),

(1.70) \[ D_i \Gamma(x) = \frac{x_i}{\sigma_n |x|^n}, \quad D_{ij} \Gamma(x) = \frac{1}{\sigma_n |x|^n} \left( \delta_{ij} - \frac{n x_i x_j}{|x|^2} \right); \]

(1.71) \[ \Delta \Gamma(x) = \sum_{i=1}^n D_{ii} u(x) = 0. \]

The choice of constants in (1.69) is motivated by the following property.

Theorem 1.37. If \( u \in C_0^2(\mathbb{R}^n) \), then

(1.72) \[ u(y) = (\Gamma * \Delta u)(y) := \int_{\mathbb{R}^n} \Gamma(y-x) \Delta u(x) \, dx \]

for all \( y \in \mathbb{R}^n \).

Proof. For fixed \( y \in \mathbb{R}^n \), choose \( r > 0 \) large enough, such that \( u \equiv 0 \) on \( \mathbb{R}^n \setminus B_r(y) \). Then in (1.68) we have \( m(r) = 0 \), and also by virtue of (1.3),

\[ \int_{B_r(y)} \Gamma_0(r) \Delta u(x) \, dx = \int_{\partial B_r(y)} \frac{\partial u}{\partial \nu} \, dS = 0. \]

Therefore, the equality (1.68) is reduced to (1.72).

Proof. For fixed \( y \in \mathbb{R}^n \), choose \( r > 0 \) large enough, such that \( u \equiv 0 \) on \( \mathbb{R}^n \setminus B_r(y) \). Then in (1.68) we have \( m(r) = 0 \), and also by virtue of (1.3),

\[ \int_{B_r(y)} \Gamma_0(r) \Delta u(x) \, dx = \int_{\partial B_r(y)} \frac{\partial u}{\partial \nu} \, dS = 0. \]

Therefore, the equality (1.68) is reduced to (1.72).

Corollary 1.38. Assume \( f \in C_0^2(\mathbb{R}^n) \). Then the function \( u := \Gamma * f \) belongs to \( C^2(\mathbb{R}^n) \), and \( \Delta u = f \) in \( \mathbb{R}^n \).
Proof. Since \( f(x) \) has compact support and \( \Gamma(x) \) is locally integrable in \( \mathbb{R}^n \), one can differentiate the equality

\[
u(y) := \Gamma * f(y) := \int_{\mathbb{R}^n} \Gamma(x) f(y - x) \, dy
\]

twice, which implies

\[
D_{ij} u(y) = \int_{\mathbb{R}^n} \Gamma(x) D_{ij} f(y - x) \, dy =: \Gamma * D_{ij} f(y) \in C(\mathbb{R}^n),
\]
i.e. \( u \in C^2(\mathbb{R}^n) \). By the previous theorem, \( \Delta u = \Gamma * \Delta f = f \) in \( \mathbb{R}^n \).

1.9.2. Green’s function. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with boundary \( \partial \Omega \) of class \( C^1 \). Then the unit outward normal \( \nu = \nu_y \) is well defined at each point \( y \in \partial \Omega \). In this section, we derive a general formula for solutions to the Dirichlet problem

\[
\Delta u = f \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial \Omega,
\]

where \( f \) and \( g \) are given functions. For arbitrary \( u, v \in C^2(\overline{\Omega}) \), by virtue of the divergence theorem,

\[
\int_{\Omega} \text{div} (u Dv - v Du) \, dx = \int_{\partial \Omega} (u Dv - v Du) \cdot \nu \, dS.
\]

We can rewrite this equality in the following form, which is known as the Green’s identity:

\[
\int_{\Omega} (u \Delta v - v \Delta u) \, dx = \int_{\partial \Omega} \left( u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) \, dS.
\]

In particular, if \( u = v = 0 \) on \( \partial \Omega \), then

\[
\int_{\Omega} u \Delta v \, dx = \int_{\Omega} v \Delta u \, dx.
\]

Definition 1.39. Suppose that for each \( x \in \Omega \), there is a function \( h^x = h^x(y) \in C^2(\overline{\Omega}) \) satisfying

\[
\Delta_y h^x(y) = 0 \quad \text{in} \quad \Omega, \quad h^x(y) = \Gamma(y - x) \quad \text{for all} \quad y \in \partial \Omega.
\]

Then the function

\[
G(x, y) := \Gamma(y - x) - h^x(y), \quad x \in \Omega, \quad y \in \overline{\Omega}, \quad x \neq y,
\]

is called the Green’s function for the domain \( \Omega \).

Note that the Green’s function \( G(x, y) \) satisfies

\[
\Delta_y G(x, y) = 0 \quad \text{in} \quad \Omega \setminus \{x\}; \quad G(x, y) = 0 \quad \text{for all} \quad x \in \Omega, \quad y \in \partial \Omega.
\]
Theorem 1.40. If $u \in C^2(\Omega)$, then

$$u(x) = \int_{\Omega} G(x,y) \Delta u(y) \, dy + \int_{\partial\Omega} \frac{\partial G(x,y)}{\partial \nu_y} u(y) \, dS_y$$

for all $x \in \Omega$.

**Proof.** Fix $x \in \Omega$ and choose a function $\zeta \in C^\infty(\Omega)$ such that $\zeta \equiv 1$ on $B_\varepsilon(x)$ for some $\varepsilon > 0$. Then the function $u$ is represented in the form $u = u_1 + u_2$, where

$$u_1 := \zeta u \in C^2_0(\Omega), \quad u_1 \equiv u \quad \text{on} \quad B_\varepsilon(x);$$

$$u_2 := (1 - \zeta)u \equiv u \quad \text{near} \quad \partial\Omega, \quad u_2 \equiv 0 \quad \text{on} \quad B_\varepsilon(x).$$

By Theorem 1.37,

$$u(x) = u_1(x) = \Gamma * \Delta u_1(x) := \int_{\Omega} \Gamma(y - x) \Delta u_1(y) \, dy.$$ 

Moreover, since $u_1$ vanishes near $\partial\Omega$, and the function $h^x$ is harmonic in \( \Omega \), from the Green’s identity (1.74) it follows

$$\int_{\Omega} h^x \Delta u_1 \, dy = \int_{\Omega} u_1 \Delta h^x \, dy + \int_{\partial\Omega} \left( h^x \frac{\partial u_1}{\partial \nu} - u_1 \frac{\partial h^x}{\partial \nu} \right) \, dS = 0.$$ 

This gives us

$$u(x) = \int_{\Omega} G(x,y) \Delta u_1(y) \, dy = I_1 + I_2,$$

where

$$I_1 := \int_{\Omega} G(x,y) \Delta u(y), \quad I_2 := -\int_{\Omega} G(x,y) \Delta u_2(y).$$

Since $u_2 \equiv 0$ on $B_\varepsilon(x)$ and $\Delta_y G(x,y) = 0$ on $\Omega \setminus B_\varepsilon(x)$, we have

$$I_2 = \int_{\Omega \setminus B_\varepsilon(x)} \left[ u_2(y) \Delta_y G(x,y) - G(x,y) \Delta u_2(y) \right] \, dy$$

$$= \int_{\partial\Omega} \left[ u_2(y) \frac{\partial G(x,y)}{\partial \nu_y} - G(x,y) \frac{\partial u_2(y)}{\partial \nu_y} \right] \, dS_y.$$ 

The boundary $\partial\Omega_\varepsilon$ consists of the exterior part $\partial\Omega$ and the interior part $\partial B_\varepsilon(x)$. In the last surface integral, the integral function vanishes on $\partial B_\varepsilon(x)$ together with $u_2$, and $G = 0$ on $\partial\Omega$. Therefore,

$$I_2 = \int_{\partial\Omega} u_2(y) \frac{\partial G(x,y)}{\partial \nu_y} \, dS_y = \int_{\partial\Omega} u(y) \frac{\partial G(x,y)}{\partial \nu_y} \, dS_y,$$

and the representation (1.78) for $u(x) = I_1 + I_2$ follows. \[\Box\]
Theorem 1.41. The Green’s function $G(x, y)$ satisfies

\begin{equation}
G(x, y) = G(y, x) \quad \text{for all} \quad x, y \in \Omega, \ x \neq y.
\end{equation}

Proof. Fix $x, y \in \Omega, \ x \neq y$, and a constant $\varepsilon \in (0, |x - y|/2)$. Let $u^\varepsilon, v^\varepsilon$ be functions in $C^2(\Omega)$, such that

\begin{align*}
    u^\varepsilon &= G(x, z) \quad \text{on} \quad \Omega \setminus B_\varepsilon(x), \\
    v^\varepsilon &= G(y, z) \quad \text{on} \quad \Omega \setminus B_\varepsilon(y).
\end{align*}

By virtue of (1.78),

\[ G(x, y) = u^\varepsilon(y) = \int_\Omega G(y, z) \Delta u^\varepsilon(z) \, dz. \]

Note that $\Delta u^\varepsilon(z) = \Delta_z G(x, z) = 0$ on the set $\Omega \setminus B_\varepsilon(x)$, which contains the ball $B_\varepsilon(y)$. Therefore,

\[ G(x, y) = \int_\Omega v^\varepsilon(z) \Delta u^\varepsilon(z) \, dz, \]

and quite similarly,

\[ G(y, x) = v^\varepsilon(x) = \int_\Omega G(x, z) \Delta v^\varepsilon(z) \, dz = \int_\Omega u^\varepsilon(z) \Delta v^\varepsilon(z) \, dz. \]

Since $u^\varepsilon = v^\varepsilon = 0$ on $\partial \Omega$, the desired equality $G(x, y) = G(y, x)$ now follows from (1.75). \qed